Chapter 6

Comparing Definitions

In this chapter we analyze the pros and cons of each of the definitions proposed to extend the concept of $\gamma-$active constraints to convex semi infinite programming. We also mention the assumptions necessary to use each one of them and show a direct relationship between them, mainly that the set of $\gamma-$active constraints defined by 4.1.2 is containted in the set defined by 5.1.1.

In Definition 4.1.2 we use the subdifferentials of the convex functions in the inequality system. This definition is a more intuitive extension of the LSIP definition of $\gamma-$active constraints. It readily provides a geometric interpretation of what is going on and avoids the use of profound convexity and closedness properties in their proofs. Another advantage of this approach is that the only assumption on the functions f_t of the inequality system is that they are to be convex and $\operatorname{int} \operatorname{dom} f_t \neq \emptyset$ for all $t \in T$, unlike the case of our second approach where we must also assume that the functions are closed (lower semi-continuous). Although, this assumption is satisfied for the type of problems that are studied in this thesis (this comes from the fact that $\operatorname{dom} f_t = \mathbb{R}^n$ for all $t \in T$) it is not satisfied for all CSIP problems. To illustrate this we present the following example of an inequality system where none of the functions f_t are closed.

Example 6.0.1 *Define $f_{\overline{x}}$ where $\overline{x} \in \mathbb{R}^2$ as follows*

$$f_{\overline{x}}(x) = \left\{ \begin{array}{c} 0 \quad \textit{if } ||x - \overline{x}|| < 1 \\ \infty \quad \textit{otherwise} \end{array} \right\}$$

It is easy to see that $f_{\overline{x}}$ is the indicator function of the set $C_{\overline{x}} = \{x \in \mathbb{R}^2 |\ ||x - \overline{x}|| < 1\}$. $f_{\overline{x}}$ is a convex function, however, it is not closed for any $\overline{x} \in \mathbb{R}^2$. Hence if we define the following inequality system $\sigma := \{f_{\overline{x}}(x) \leq 0\}$ where $\overline{x} \in \{(0,1),(0,0)\}$, we will not be able to obtain the same results using Definition 5.1.1 because $f_{\overline{x}}^{**} \neq f_{\overline{x}}$.

On the other hand, using this first approach, we obtain a counterexample for the crucial inclusion $D(F, \overline{x})^0 \subseteq \operatorname{cl}\operatorname{cone} W_\partial\,(\overline{x}, \gamma)$ and to the convex equivalent of Proposition 3.5.1 (see Example 4.4.1). However, using the second approach, by means of Definition 5.1.1, we obtain the inclusion $D(F, \overline{x})^0 \subseteq \operatorname{cl}\operatorname{cone} W_L\,(\overline{x}, \gamma)$ and hence extend Proposition 3.5.1 to the problems studied in this thesis. In this approach, we sacrifice the intuitive geometric interpretation that Definition 4.1.2 provides but we take advantage of an important characteristic of the problems to be studied: closedness of the f_t for all $t \in T$. This characteristic comes as a result of each of the functions f_t having $\operatorname{dom} f_t = \mathbb{R}^n$.

Another noteworthy characteristic about the two approaches is that the set of $\gamma-$active constraints defined by 4.1.2 is a subset of the set defined by 5.1.1. The following proposition formalizes this fact.

Proposition 6.0.4 *Given $\overline{x} \in \mathbb{R}^n$ and $\gamma > 0$, $W_\partial(\overline{x}, \gamma) \subseteq W_L(\overline{x}, \gamma)$.*

Proof. Suppose $g \in W_\partial(\overline{x}, \gamma)$. Therefore there exists $y \in \overline{x} + \gamma B_n$ and $s \in T(y)$ such that $g \in \partial f_s(y)$. From Fenchel's inequality for conjugate convex functions we know that $g'y \leq f_s(y) + f_s^*(g)$. Since $g \in \partial f_s(y)$ the inequality holds as equality and also since $s \in T(y)$ we have that

$$\begin{aligned} g'y &= f_s(y) + f_s^*(g) \\ g'y &= f_s^*(g) \end{aligned}$$

Therefore $g \in W_L(\overline{x}, \gamma)$. ■

Remark 6.0.1 *We now refer to Example 4.4.1 as a counterexample to show that the reverse inclusion does not hold. Recall that at* $\overline{x} = (0, 0)$ *and* $\gamma > 0$, $W_\partial(\overline{x}, \gamma) = \{0, 0\}$. *However, under the new definition* $W_L(\overline{x}, \gamma) = \mathbb{R}^n$ *for any* $\gamma > 0$. *Hence* $W_L(\overline{x}, \gamma) \nsubseteq W_\partial(\overline{x}, \gamma)$.

As we can see, each approach has its advantage and disadvantage. The selection of an approach will depend on the details of the CSIP problem, hence the reason we include both approaches in this document.