Chapter 3

Preliminaries

In this chapter we review the basic concepts and ideas used in the theorems and proofs in this thesis. First, topology concepts and definitions from [25], [4] and [18] are presented, followed by a brief overview of differential calculus in $\mathbb{R}^n$ and convex analysis concepts, definitions and theorems from [27] , [31] and [2]. Then we present a brief review of the definition and properties of active constraints in the linear programming context. We then conclude by presenting the definition of $\gamma-$active constraints in Linear Semi-Infinite Programming and enunciate the properties proven in [10].

3.1 Topology

Definition 3.1.1 *A topology on a set X is a collection τ of subsets of X having the following properties*

i) ϕ and X are in τ

ii) The union of any subcollection of τ is in τ.

iii) The intersection of any finite subcollection of τ is in τ.

Remark 3.1.1 *A topological space is an ordered pair (X, τ) consisting of a set X and its topology τ. Any element $A \in \tau$ is referred to as an open set. A set $B \in \tau$ is closed if its complement in X is open.*

Next, some of the topological characteristics of a given subset of X are defined.

Definition 3.1.2 *Given a subset A of a topological space (X, τ),*

i) A is a neighbourhood of a point $x \in X \Leftrightarrow A$ contains an open set to which x belongs to.

ii) The interior of A denoted as $\operatorname{int} A$ is defined as the union of all open sets contained in A.

iii) The closure of A denoted as $\operatorname{cl} A$ is defined as the intersection of all closed sets containing A.

iv) The boundary of A denoted as $\operatorname{bd} A$ is defined as $\operatorname{cl} A \setminus \operatorname{int} A$.

Definition 3.1.3 *Given a set $S \subseteq \mathbb{R}^n$, the affine hull, denoted as $\operatorname{aff}(S)$ is defined as*

$$\operatorname{aff}(S) = \{\sum_{i=1}^{k} \alpha_i x_i | \alpha_i \in \mathbb{R}, \sum_{i=1}^{k} \alpha_i = 1 \text{ and } x_i \in S\}$$

Definition 3.1.4 *Given a convex set $S \subseteq \mathbb{R}^n$ its relative interior, denoted as $\operatorname{ri} S$ is the interior which results when S is regarded as a subset of its affine hull (i.e. $\operatorname{ri} S = \{x \in \operatorname{aff} S | \ \exists \varepsilon > 0, \ (x + \varepsilon B_n) \cap (\operatorname{aff} S) \subset S\}$)*

Theorem 3.1.1 *The intersection of an arbitrary number of closed sets is closed and the union of a finite number of closed sets is closed.*

These definitions are used in establishing the topologic relations between a point $x \in \mathbb{R}^n$ and the feasible set F of a given inequality system σ.

3.2 Calculus

Before elaborating on the study of convex analysis, we first present some of the basic calculus concepts of functions $f : S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ such as $f(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2$ (see Figure 3.1).

Definition 3.2.1 *f is continuous at $a \in \mathbb{R}^n$ if $\forall \varepsilon > 0, \exists \delta > 0$ such that $||x - a|| < \delta \Longrightarrow |f(x) - f(a)| < \varepsilon$*

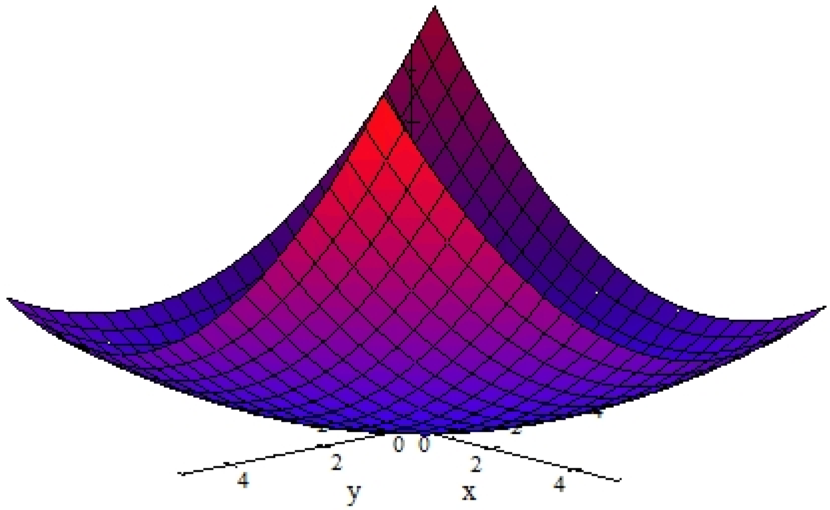

Figure 3.1: Graph of $f(x, y) = x_1^2 + x_1x_2 + x_2^2$

Remark 3.2.1 *Any function that is differentiable at a point $x \in \mathbb{R}^n$ is continuous at that point, however continuity at a point does not imply the differentiablility of the function at that point. An example of such a function is $f(x) = |x|$ (see Figure 3.2) which is continuous at $x = 0$ but it is not differentiable at that point.*

Definition 3.2.2 *The partial derivative of f with respect to the coordinate i at a point $a = (a_1, a_2, ..., a_n)$ is defined as*

$$\frac{\partial f}{\partial x_i}(a_1, a_2, ..., a_n) = \lim_{\delta \to 0^+} \frac{f(a_1, ..., a_i + \delta, ..., a_n) - f(a_1, a_2, ..., a_n)}{\delta} \tag{3.1}$$

Definition 3.2.3 *The gradient of a function f at a point $x \in \operatorname{dom} f$ is defined as*

$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, ..., \frac{\partial f(x)}{\partial x_n} \right) \tag{3.2}$$

Definition 3.2.4 *The directional derivative of a function f with respect to a direction d at a point x is defined as*

$$f'(x, d) = \lim_{\delta \to 0^+} \frac{f(x + \delta d) - f(x)}{\delta} \tag{3.3}$$

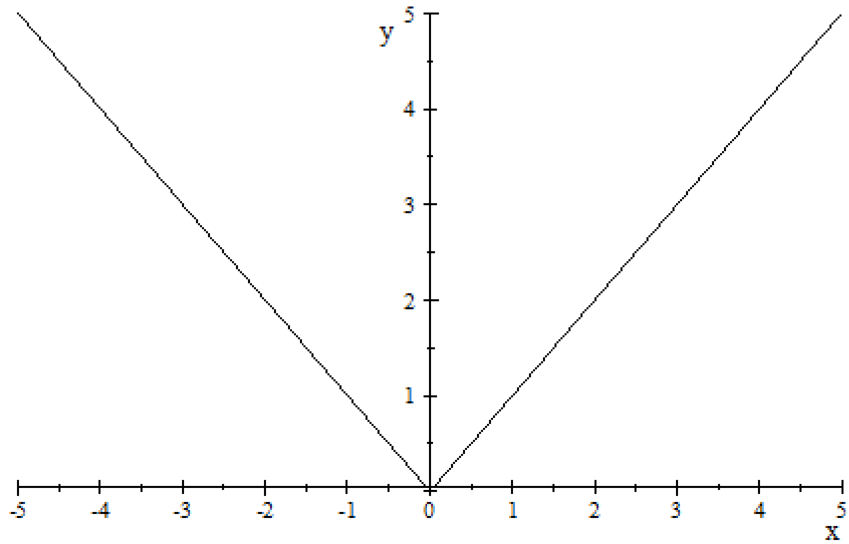

Figure 3.2: Graph of $f(x) = |x|$

Remark 3.2.2 *The partial derivative* $\frac{\partial f(x)}{\partial x_i}$ *is the directional derivative of a function* f *in the direction of the vector* $(0, 0, ...1, ..., 0)$ *where* 1 *is at the* i*th position of the vector.*

Definition 3.2.5 *Let* $f : S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ *be a two time differentiable function, therefore the Hessian Matrix of* f *at* $x = (x_1, x_2, ..., x_n)$ *is defined as*

$$H(f, x) := \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 x_n} \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial x_n x_1} & \frac{\partial^2 f}{\partial x_n x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

3.3 Convex Analysis

Given that the feasible set $F \subseteq \mathbb{R}^n$ and the functions f_t in the problems presented are convex for all $t \in T$, it is necessary to review some of the concepts from convex analysis. This section is dedicated to present basic definitions of convex sets, cones, half spaces, convex hulls and convex functions in $\mathbb{R}^n$. Theorems that will be used later on are enunciated, a few of them with their corresponding proofs. The reader may consult [27] ,[1], [31], [16] and [2] for the proofs of the corresponding theorems.

Definition 3.3.1 *A set $C \subseteq \mathbb{R}^n$ is convex if $(\lambda x + (1-\lambda)y) \in C$ for all $x, y \in C$ and $\lambda \in [0,1]$.*

Definition 3.3.2 *The convex hull of a set $S \subseteq \mathbb{R}^n$, denoted as $\operatorname{conv} S$ is the intersection of all convex sets in $\mathbb{R}^n$ that contain S.*

Definition 3.3.3 *For any non zero vector $b \in \mathbb{R}^n$ and any $\beta \in \mathbb{R}$ the sets*

$$\{x \in \mathbb{R}^n | b'x \leq \beta\}, \ \ \{x \in \mathbb{R}^n | b'x \geq \beta\}$$

are called closed half-spaces. The sets

$$\{x \in \mathbb{R}^n | b'x < \beta\}, \ \ \{x \in \mathbb{R}^n | b'x > \beta\}$$

are called open half-spaces.

Definition 3.3.4 *A set $S \subseteq \mathbb{R}^n$ is an affine set if given $x_1, x_2, ...x_m \in S$ and $\lambda_1, \lambda_2 ... \lambda_m \in \mathbb{R}$ such that $\displaystyle\sum_{i=1}^{m} \lambda_i = 1$ then $\displaystyle\sum_{i=1}^{m} \lambda_i x_i \in S$*

Theorem 3.3.1 *The intersection of any arbitrary set of convex sets is convex.*

The following definitions provide a characterization that is useful in Linear Programming (LP) problems.

Definition 3.3.5 *A set $C \subseteq \mathbb{R}^n$ that can be expressed as the intersection of finitely many closed half spaces in $\mathbb{R}^n$ is called a polyhedral convex set.*

Definition 3.3.6 *A point $x \in \mathbb{R}^n$ is an extreme point of a convex set C if $C \backslash \{x\}$ is convex.*

Next we define the concept of a convex function. We present three equivalent definitions of a convex function. The first based on a geometric characterization known as the epigraph, the second based on the convex combination of two points and the last based on a generalized convex combination of n points. Before presenting these we must first define the concept of the epigraph of a function.

Definition 3.3.7 *Let $f : S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$,the epigraph of f is defined as*

$$\mathrm{epi}(f) = \{(x, u) \in \mathbb{R}^{n+1} |\ x \in S, u \in \mathbb{R}, u \geq f(x)\}$$

We now present the three equivalent definitions of a convex function.

Definition 3.3.8 *Given a convex set $S \neq \varnothing$, a function $f : S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is convex if* epi(f) *is a convex set in $\mathbb{R}^{n+1}$.*

Definition 3.3.9 *Given a convex set $S \neq \varnothing$, a function $f : S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is convex if* $\forall n \geq 2, \forall x_1, x_2, ..., x_n \in S, \forall \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}_+ : \sum_{i=1}^{n} \lambda_i = 1$

$$f(\sum_{i=1}^{n} \lambda_i x_i) \leq \sum_{i=1}^{n} \lambda_i f(x_i)$$

Definition 3.3.10 *Given a convex set $S \neq \varnothing$, a function $f : S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is convex if $\forall$ $x_1, x_2 \in S$, and $\lambda \in (0, 1)$*

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$$

Remark 3.3.1 *If a function f satisfies the strict inequalities presented in the definitions 3.3.9 and 3.3.10 then f is said to be strictly convex.*

The following proposition proves the convexity of the feasible set F of any given CSIP problem of the form (2.1).

Proposition 3.3.1 *Given $F = \{x \in \mathbb{R}^n | f_i(x) \leq 0, i \in I\}$ where f_i is convex for all $i \in I$ and I is an arbitrary set of indices , F is a convex set.*

Proof. Let x_1 and $x_2 \in F$ then $f_i(x_1) \leq 0$ and $f_i(x_2) \leq 0$ for all $i \in I$. Taking $\lambda \in [0, 1]$, since f_i is convex for all $i \in I$ we have

$$f_i(\lambda x_1) + (1 - \lambda)x_2) \leq \lambda f_i(x_1) + (1 - \lambda)f_i(x_2) \leq 0 \text{ for all } i \in I$$

Therefore $\lambda x_1 + (1-\lambda)x_2 \in F$ and hence F is a convex set. ■

Next we present some preliminary concepts that will be used in showing later on that the feasible set defined in the problem (2.1) is a closed set in $\mathbb{R}^n$.

Definition 3.3.11 *A convex function f is proper if its epigraph is non-empty and contains no vertical lines, i.e. $f(x) < \infty$ for at least one x and $f(x) > -\infty$ for all x.*

Definition 3.3.12 *A function f is said to be lower semi continuous at $x \in S$ if $f(x) = \liminf_{y \to x} f(y)$. A function is said to be lower semi continuous in S if it is lower semi continuous at all points in S.*

Definition 3.3.13 *Given a function $f : \mathbb{R}^n \to \mathbb{R}$, the lower semi continuous hull of f is the greatest lower semi continuous function majorized by f, namely the function whose epigraph is the closure in $\mathbb{R}^{n+1}$ of the epigraph of f.*

Now we show that the feasible defined in the problem (2.1) is a closed set in $\mathbb{R}^n$. For this we must first define the concept of the closure $\operatorname{cl} f$ of a function f then proceed to showing that the functions f_t as defined in the problem (2.1) are closed functions and through this arrive to the conclusion that the feasible set of the the problem is closed.

Definition 3.3.14 *The closure of a proper convex function f denoted by $\operatorname{cl} f$ is the lower semi continuous hull of the function f.*

Definition 3.3.15 *A function f is closed if $\operatorname{cl} f = f$.*

Next we present a corollary that justifies that all functions f_t as defined in the problem (2.1) are closed functions followed by another corollary which states that the solution set to an inequality of the form $f(x) \leq \alpha$ where f is a closed function and $\alpha \in \mathbb{R}$ is a closed set.

Corollary 3.3.1 *If f is a proper convex function such that $\operatorname{dom} f$ is an affine set (for example $\operatorname{dom} f = \mathbb{R}^n$) then f is a closed function*

Corollary 3.3.2 *If f is a closed function then the set*

$$F = \{x | f(x) \leq \alpha\}$$

is a closed set.

Having this and Theorem 3.1.1 we arrive to the conclusion that the feasible set F defined as

$$F := \{x \in \mathbb{R}^n | f_t(x) \leq 0 \quad \text{for all } t \in T\}$$

is a closed set in $\mathbb{R}^n$.

Next we present other properties of convex functions that are used in later chapters. For the following theorems and propositions we refer to $f : S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ where S is a non empty set.

Theorem 3.3.2 *If f is a proper convex function defined in S then*
i) S is a convex set.
ii)f is continuous in $\operatorname{int} S$
iii) If $\lambda \geq 0$ then λf is convex

Theorem 3.3.3 *Let $S \neq \varnothing$, if $f : S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ and $g : S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ are convex functions then $f + g$ is a convex function.*

Theorem 3.3.4 *Let $S \neq \varnothing$ and $f : S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a proper convex function. If $x^* \in S$ is a local minimizer then x^* is a global minimizer.*

Proof. Suppose x^* be a local minimizer then there $\exists$ a neighbourhood $U \subseteq \mathbb{R}^n$ such that $f(x^*) \leq f(z)$ $\forall z \in U$. Next we prove that for all $y \in S$

$$f(x^*) \leq f(y)$$

We take the convex combination $(1-\lambda)x^* + \lambda y$ where $0 \leq \lambda \leq 1$. Note that as $\lambda \longrightarrow 0$, $(1-\lambda)x^* + \lambda y \in U$ then

$$\begin{aligned} f(x^*) &\leq f((1-\lambda)x^* + \lambda y) && \text{for } \lambda \text{ sufficiently small} \\ &\leq (1-\lambda)f(x^*) + \lambda f(y) && \text{because } f \text{ is a convex function} \end{aligned}$$

manipulating this inequality we obtain $f(x^*) \leq f(y)$ for all $y \in S$ therefore x^* is a global minimizer. ■

Theorem 3.3.5 *Let $S \neq \varnothing$ and $f : S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$. If f is convex then the set of global minimizers of f, F^{opt}, is a convex set.*

Proof. Let x_1 and x_2 be global minimizers of f then $f(x_1) = f(x_2) = \min_{x \in S} f(x) \leq f(x)$ $\forall x \in S$. Taking the convex combination of x_1 and x_2 we have

$$\begin{aligned} f(\lambda x_1 + (1-\lambda)x_2) &\leq \lambda f(x_1) + (1-\lambda)f(x_2) = \\ &= \min_{x \in S} f(x) \end{aligned}$$

on the other hand, $\min_{x \in S} f(x) \leq f(\lambda x_1 + (1-\lambda)x_2)$ (by definition of $\min_{x \in S} f(x)$). Therefore we have that $f(\lambda x_1 + (1-\lambda)x_2) = \min_{x \in S} f(x)$, i.e. $\lambda x_1 + (1-\lambda)x_2$ is a global minimizer, hence the set of global minimizers is convex. ■

Theorem 3.3.6 *Let $f : S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, $f \in C^1$ be a convex function and $x^* \in S$. If $\nabla f(x^*) = 0$ then x^* is a global minimizer.*

Next we present a concept that is essential to our first approach to extending $\gamma-$active constraints to CSIP.

Definition 3.3.16 *A vector $g \in \mathbb{R}^n$ is a sub-gradient of the function f at the point x if $\forall$ $z \in \operatorname{dom} f$ the following inequality is satisfied*

$$f(z) \geq f(x) + g'(z - x) \tag{3.4}$$

The set of all the sub-gradients of f at a point x is called the sub-differential of f at x.

Remark 3.3.2 *The sub-differential of f at x denoted as $\partial f(x)$ is a non empty compact set if f is continuous at the point x.*

We present some important properties of the subdifferential of a proper convex function in $\mathbb{R}^n$.

Theorem 3.3.7 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a proper convex function then at each point $x \in \mathbb{R}^n$, $\partial f(x) \neq \emptyset$ and $\partial f(x)$ is a closed bounded convex set.*

Theorem 3.3.8 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a proper convex function. If f is differentiable at x then $\nabla f(x)$ is the unique subgradient of f at x, so that in particular*

$$f(z) \geq f(x) + \nabla f(x)'(z - x) \quad \textit{for all } z \in \mathbb{R}^n$$

Based on these properties we are able to state that given a problem of the form (2.1), the set $\partial f_t(x)$ is a non empty compact convex set for all $t \in T$ and $x \in \mathbb{R}^n$.

The use of cones, convex cones, dual cones, polar cones and recession cones are important as will be seen in future chapters. The following are the definitions of these concepts and some useful theorems.

Definition 3.3.17 *A set $K \subseteq \mathbb{R}^n$ is a cone if it is closed under positive scalar multiplication (i.e. $\lambda x \in K$ for all $x \in K$, $\lambda > 0$).*

Remark 3.3.3 *A convex cone is a cone that is closed under convex combinations.*

Theorem 3.3.9 *The intersection of any arbitrary set of convex cones is convex.*

Definition 3.3.18 *The convex cone generated by an arbitrary set $S \subseteq \mathbb{R}^n$ denoted as $\operatorname{cone} S$ is the convex cone obtained by adjoining the origin to the smallest convex cone containing S (i.e. $\operatorname{cone} S = \{\lambda s |\ \lambda > 0,\ s \in \operatorname{conv} S\} \cup \{0_n\}$).*

Definition 3.3.19 *Given a set* $S \subseteq \mathbb{R}^n$

i) The asymptotic cone of S *is defined as the set of limits of the form* $\lim_{k\to\infty} \lambda_k x_k$ *where* $\lambda_k \in \mathbb{R}_+,\ x_k \in S,\ k = 1, 2, .., \lambda_k \downarrow 0$.

Definition 3.3.20 *Given a convex cone* $K \subseteq \mathbb{R}^n$

i)The dual K^* *of the cone* K *is defined as*

$$K^* = \{x \in \mathbb{R}^n | k'x \geq 0 \textit{ for all } k \in K\}$$

iii) The polar K^o *of* K *is defined as*

$$K^o = \{x \in \mathbb{R}^n | k'x \leq 0 \textit{ for all } k \in K\}$$

Remark 3.3.4 *Another important property of the polar of a cone* K *are that the polar* K^{oo} *of* K^o *is* $\operatorname{cl} K$*, see [27] (the same applies for the dual cone* K^**). Another important property of the polar of cones is the following.*

Theorem 3.3.10 *Let* K*,* $G \subseteq \mathbb{R}^n$ *be convex cones. If* $K^o \subseteq G$ *then* $G^o \subseteq \operatorname{cl} \operatorname{cone} K$

Remark 3.3.5 *An important property of the polar and dual of a convex cone is the fact that since they are both closed sets (by definition) we then have that given a convex cone* K*,* $(\operatorname{cl} K)^o = K^o$ *and* $(\operatorname{cl} K)^* = K^*$*.*

Next we present the definitions as given in [1] of the tangent and normal cones of a given subset $S \subseteq \mathbb{R}^n$.

Definition 3.3.21 *Given* $S \subseteq \mathbb{R}^n$ *and a vector* $x \in S$*, a vector* y *is said to be tangent to* S *at* x *if either* $y = 0$ *or there exists a sequence* $\{x_k\} \subset S$ *such that* $x_k \neq x$ *for all* k *and* $x_k \longrightarrow x$*,* $\frac{x_k - x}{||x_k - x||} \longrightarrow \frac{y}{||y||}$*. The set of all tangents of* S *at* x *is called the tangent cone of* S *at* x *denoted by* $T_S(x)$*.*

Definition 3.3.22 *Given $S \subseteq \mathbb{R}^n$ and a vector $x \in S$, a vector z is said to a normal of S at x if there exist sequences $\{x_k\} \subset S$ and $\{z_k\}$ such that $x_k \longrightarrow x$, $z_k \longrightarrow z$, $z_k \in T_S(x_k)^o$ for all k . The set of all normals of S at x is called the normal cone of S at x denoted by $N_S(x)$.*

Next we present properties with respect to the tangent cone $T_F(x)$ and normal cone $N_F(x)$ of a convex set F at a point $x \in F$.

Proposition 3.3.2 *The tangent cone $T_F(x)$ of a closed convex set F at x is the closure of the cone of feasible directions of F at x:*

$$T_F(x) = \operatorname{cl} D(F, x)$$

Proposition 3.3.3 *The normal cone $N_F(x)$ of a closed convex set F at x is the negative polar to the tangent cone $T_F(x)$.*

The following proposition is a well known result in convex analysis that characterizes optimality of a point in the feasible set of a given Convex Programming problem by means of the normal cone of the feasible set.

Theorem 3.3.11 *Let $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a convex function which is to be minimized over a closed convex set F. Given $\overline{x} \in F$, $\overline{x}$ minimizes h over F if and only if $0_n \in \partial h(\overline{x}) + N_F(\overline{x})$, i.e. there exists $\widehat{h} \in \partial h(\overline{x})$ such that $-\widehat{h} \in N_F(\overline{x})$.*

A proof of proposition 3.3.11 can be found in [16] page 294.

Next we present an important tool used in the second definition of $\gamma-$active constraints presented in this thesis as an extension of the linear case.

Definition 3.3.23 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ then its conjugate f^* is defined as*

$$f^*(u) := \sup_{x \in \mathbb{R}^n} \{u'x - f(x)\}$$

This definition can be interpreted in several ways, and is particular in the case of convex functions since its existence arises from the fact that the epigraph of a convex function is convex in $\mathbb{R}^{n+1}$. It is actually the pointwise supremum of the affine functions of the form $g(x^*) = x'x^* - u$ such that (x, u) belongs to $\operatorname{epi} f \subseteq \mathbb{R}^{n+1}$. An important property to note about f^* is that it is a closed convex function. Next we enunciate some other important properties of f^* as presented in [27].

Theorem 3.3.12 *Let f be a convex function. The conjugate function f^* is then a closed convex function, proper if and only if f is proper. Moreover,* $(\operatorname{cl} f)^* = f^*$ *and* $f^{**} = \operatorname{cl} f$.

This theorem is very important since it will provide us with the base of our new definition of $\gamma-$active constraints, more specifically the fact that $f^{**} = \operatorname{cl} f$ which in our case becomes $f^{**} = f$ since the functions f_t are closed proper convex functions for all $t \in T$.

The following theorem plays a very important role in the proof of one of the feasible set theorems in Chapter 5. Let $f'(x, d)$ denote the directional derivative of f at x in the direction d and $\partial f(x)$ be the set of all subgradients of f at the point x.

Theorem 3.3.13 *Given a proper convex function $f : \mathbb{R}^n \to \mathbb{R}$ then any point $x \in \operatorname{int}(\operatorname{dom} f)$ and any direction $d \in \mathbb{R}^n$ satisfy*

$$f'(x, d) = \max\{g'd |\ g \in \partial f(x)\}$$

It is common in optimization to study certain characteristics of the inequality system that defines the feasible set. These characteristics known as Constraint Qualifications often provide sufficient conditions for the existence and uniqueness of solutions to the optimization problem (see [23] and [22]). The following is a well known constraint qualification.

Definition 3.3.24 *The inequality system σ satisfies the classical* Slater condition *if there exists $x \in \mathbb{R}^n$ such that $f_t(x) < 0$, for all $t \in T$*

3.4 Active Constraints

Active constraints is a concept that is used in many branches of optimization theory. They are used to determine the topological relation between a given point $x \in \mathbb{R}^n$ and the solution set F of a system of inequalities σ. Depending on the number of active constraints at a given point $x \in \mathbb{R}^n$ we can discover whether it is an interior point, exterior point, or boundary point. Active constraints can also be used to determine the recession directions, and thus the recession cone of a given system of inequalities σ. They also determine the feasible directions of a given point $x \in F \subseteq \mathbb{R}^n$ with respect to σ. The following are the definitions and a proposition that allow us to use active constraints for the purposes previously stated.

We begin by defining feasible directions and the cone of feasible directions.

Definition 3.4.1 *Given $\overline{x} \in F$, a vector $d \in \mathbb{R}^n$ is a feasible direction at $\overline{x}$ if $\overline{x} + \lambda d \in F$ for some $\lambda > 0$.*

Definition 3.4.2 *Let $F \subseteq \mathbb{R}^n$. We define the set of feasible directions at the point $\overline{x} \in F$ as follows*

$$D(F, \overline{x}) := \{d \in \mathbb{R}^n \mid \overline{x} + \lambda d \in F, \text{ for a certain } \lambda > 0\}.$$

Obviously $D(F, \overline{x}) = \mathbb{R}^n$ if $\overline{x}$ is an interior point of F.

Next we define active constraints for the linear inequality systems of LSIP problems of the form 2.3.

Definition 3.4.3 *Given $\overline{x} \in F$ the set of active constraints at $\overline{x}$ is defined as*

$$T(\overline{x}) := \{t \in T \mid a_t \overline{x} = b\}$$

An important set used to establish a relationship between $T(\overline{x})$ and $D(F, \overline{x})$ is

$$A(\overline{x}) = \text{cone}\{a_t \in \mathbb{R}^n \mid t \in T(\overline{x})\}$$

Next is a proposition found in [10] that summarizes the usefulness of active constraints in linear programming. In general, given $\overline{x} \in F$ we have that $\text{cl}\, A(\overline{x}) \subseteq D(F, \overline{x})^*$, however there exists a class of systems that satisfy a bit more than the inclusion.

Definition 3.4.4 *An optimization problem P is said to be Locally Farkas Minkowski at $\overline{x} \in F$ if $A(\overline{x}) = D(F, \overline{x})^*$.*

Proposition 3.4.1 *Given $\overline{x} \in F$, the following statements hold:*

(i) $F = \{\overline{x}\}$ if and only if $0_n \in \operatorname{int} D(F, \overline{x})^*$.

(ii) $\overline{x} \in F^{opt}$ if and only if $c \in D(F, \overline{x})^*$.

(iii) If $\dim D(F, \overline{x})^* = n$, then $\overline{x} \in \operatorname{extr} F$.

These concepts and definitions form the basis of the theory of $\gamma-$active constraints in LSIP problems and will be referred to in later chapters.

3.5 γ-Active Constraints in Linear Semi-Infinite Programming

The definition of $\gamma-$active constraints in LSIP along with some lemmas, propositions and their respective proofs are presented in this section. The concepts and theory that are presented in this section come from [30] for LSIP problems whose feasible set F is defined by a linear inequality system σ of the following form

$$\sigma = \left\{ a_t' x \geq b_t, t \in T \right\}. \tag{3.5}$$

Definition 3.5.1 *Given $\overline{x} \in F$ and $\gamma > 0$, we define the set of γ -active constraints at $\overline{x} \in \mathbb{R}^n$ as*

$$W(\overline{x}, \gamma) := \left\{ a_t \mid t \in T \text{ and } a_t' y = b_t \text{ for a certain } y \in \overline{x} + \gamma B_n \right\}.$$

In other words, if $a_t \neq 0_n$, then $a_t \in W(\overline{x}, \gamma)$ if and only if $a_t'\overline{x} < b_t + \gamma \|a_t\|$. Obviously, $\{a_t | \, t \in T(\overline{x})\} \subset W(\overline{x}, \gamma)$. Moreover, if $\overline{x} \in \operatorname{int} F$ there will exist $\gamma_0 > 0$ sufficiently small such that $W(\overline{x}, \gamma) \setminus \{0_n\} = \emptyset$ for all γ such that $0 < \gamma < \gamma_0$. Next we enunciate the propositions and lemmas presented in [30] without their respective proofs.

The following lemma provides basic characteristics of the definition presented above.

Lemma 3.5.1 *Given $\overline{x} \in \operatorname{bd} F$, the following statements hold:*

(i) $W(\overline{x}, \gamma)$ *contains at least a nonzero vector for all* $\gamma > 0$.

(ii) If $T(\overline{x}) = \emptyset$, *then* $W(\overline{x}, \gamma)$ *is an infinite set for all* $\gamma > 0$.

(iii) If $|T| < \infty$, *then* $W(\overline{x}, \gamma) = \{a_t,\ t \in T(\overline{x})\}$ *for* $\gamma > 0$ *sufficiently small.*

The following lemmas show that the $\gamma-$active constraints at $\overline{x} \in F$ allow us to check the feasibilty of points in the open ball $\overline{x} + \gamma B_n$ and of given directions at $\overline{x}$.

Lemma 3.5.2 *Let* $\overline{x} \in F$ *and* $y \in \overline{x} + \gamma B_n$, $\gamma > 0$. *Then* $y \in F$ *if and only if* $a_t' y \geq b_t$ *for all* $a_t \in W(\overline{x}, \gamma)$.

Lemma 3.5.3 *Let* $\overline{x} \in F$ *and* $d \in \mathbb{R}^n$. *The following statements are true:*

(i) If for a certain $\gamma > 0$ *we have* $a_t' d \geq 0$ *for all* $a_t \in W(\overline{x}, \gamma)$, *then* $d \in D(F, \overline{x})$. *So* $D(F, \overline{x})^* \subset \operatorname{cl} \operatorname{cone} W(\overline{x}, \gamma)$ *for all* $\gamma > 0$.

(ii) If $d \in D(F, \overline{x})$ *and* $|T| < \infty$, *then there exists some* $\gamma_0 > 0$ *such that* $a_t' d \geq 0$ *for all* $a_t \in W(\overline{x}, \gamma)$ *and all positive* $\gamma < \gamma_0$. *In such a case,* $D(F, \overline{x})^* = \operatorname{cone} W(\overline{x}, \gamma)$.

The following proposition provides necessary conditions for optimality and for certain characteristics of the feasible set.

Proposition 3.5.1 *Given* $\overline{x} \in F$ *and* $\gamma > 0$, *the following statements hold:*

(i) If $F = \{\overline{x}\}$, *then* $0_n \in \operatorname{int} \operatorname{cone} W(\overline{x}, \gamma)$.

(ii) If $\overline{x} \in F^{opt}$, *then* $c \in \operatorname{cl} \operatorname{cone} W(\overline{x}, \gamma)$.

(iii) If $\overline{x} \in \operatorname{extr} F$, *then* $\dim \operatorname{cone} W(\overline{x}, \gamma) = n$.

The following proposition shows that the conditions presented in Proposition 3.5.1 are both necessary and sufficient in the Finite Linear Programming Case.

Proposition 3.5.2 *Let* $\overline{x} \in F$ *and* $|T| < \infty$. *The following statements hold:*

(i) If $0_n \in \operatorname{int} \operatorname{cone} W(\overline{x}, \gamma)$ *for all* $\gamma > 0$, *then* $F = \{\overline{x}\}$.

(ii) If $c \in \operatorname{cl} \operatorname{cone} W(\overline{x}, \gamma)$ *for all* $\gamma > 0$, *then* $\overline{x} \in F^{opt}$.

(iii) If $\dim \operatorname{cone} W(\overline{x}, \gamma) = n$ *for all* $\gamma > 0$, *then* $\overline{x} \in \operatorname{extr} F$.

These definitions and theory have been studied only in the LSIP case, in the following chapters, these concepts will be extended to the CSIP case with proofs that will hold for the both the CSIP and the LSIP case.