Chapter 6

Examples

6.1 Example 1

This example is an infinite-dimensional extension of Example 4.2 found in [9].

Let $X = L_2[0,1] \times L_2[0,1]$ and $Y = \mathbb{R}$, with our programming problem defined as follows:

$$P_1 : \min\{f(x)|g(x) \leq 1\}$$

where $f, g : X \to Y$ are defined by

$$\begin{aligned} f(x) &= -\int_0^1 x_1 x_2 dt \\ g(x) &= \int_0^1 (x_1 + 4x_2) dt \end{aligned}$$

with

$$M = \{x \in X | \int_0^1 (x_1 + 4x_2) dt \leq 1\}$$

and the optimal $\bar{x}$ with its associated Langrange multiplier l are:

$$\bar{x} \overset{a.e.}{=} (\tfrac{1}{2}, \tfrac{1}{8}) \qquad\qquad l = \tfrac{1}{8}.$$

Notice that the second Fréchet derivative of the Lagrangian at $\bar{x}$

$$L_p''(x,l_p)(h,h) = -2\int_0^1 h_1(t)h_2(t)dt$$

is not positive definite on X.

Also, since there exist $x \in M$ for which $f(x) \not> 0$ we will apply the exponential transformation before we can define the p-power formulation. Then, our first equivalent problem is

$$P_1^{eq} : \min\{e^{f(x)} | e^{g(x)} \leq e\}$$

and our equivalent problem using the p-power formulation is

$$P_1^p : \min\{(e^{f(x)})^p | (e^{g(x)})^p \leq e^p\}$$

with its associated p-power Lagrangian function given by

$$L_p(x,l_p) = (e^{f(x)})^p + l_p((e^{g(x)})^p - e^p).$$

The first derivative of the Lagrangian function will be defined as

$$L_p'(x,l_p)h = \frac{d}{d\alpha}[(e^{f(x+\alpha h)})^p]\big|_{\alpha=0} + \frac{d}{d\alpha}[\langle l_p, (e^{g(x+\alpha h)})^p - e^p\rangle]\big|_{\alpha=0}$$

which for our particular $f(x)$ and $g(x)$ gives us

$$L_p'(x,l_p)h = -p(e^{f(x)})^p \int_0^1 (x_1h_2 + x_2h_1)dt + l_p p(e^{g(x)})^p \int_0^1 (h_1 + 4h_2)dt.$$

We now wish to find the Lagrange multiplier for the p-power formulation corresponding to the optimal point, which we already know is $\bar{x} \overset{a.e.}{=} (\frac{1}{2}, \frac{1}{8})$, then from (5.6) we get

$$L_p'(\bar{x},l_p)h = (l_p - \frac{1}{8}e^{-\frac{p}{16}})\int_0^1 (\frac{1}{4}h_1 + h_2)dt = 0$$

where we can easily see that $l_p = \frac{1}{8}e^{-\frac{p}{16}}$.

Our next step is to calculate the second Fréchet derivative for the Lagrangian function

$$L_p''(x, l_p)(h, h) = \frac{d}{d\alpha} L_p'(x + \alpha h, l_p)h\big|_{\alpha=0}$$

$$L_p''(x, l_p)(h, h) = -p(e^{f(x)})^p[\int_0^1 2(h_1h_2)dt - p[\int_0^1 (x_1h_2 + x_2h_1)dt]^2] + l_p p^2(e^{g(x)})^p[\int_0^1 (h_1 + 4h_2)dt]^2$$

and evaluate it at the optimal point $\bar{x}$ and its associated multiplier l_p

$$L_p''(\bar{x}, l_p)(h, h) = pe^{-\frac{p}{16}}[\frac{1}{64}p(1 + 8e^p)[\int_0^1 (h_1 + 4h_2)dt]^2 - 2\int_0^1 h_1h_2dt].$$

Here we can see that $pe^{-\frac{p}{16}} > 0$, then for this second Fréchet derivative to be positive definite, we need

$$\frac{1}{128}p(1 + 8e^p)[\int_0^1 (h_1 + 4h_2)dt]^2 > \int_0^1 h_1h_2dt.$$

It is clear that when $p \to \infty$, $(1 + 8e^p) \to \infty$. Then, there exists a sufficiently large p for which the inequality holds for all $h \in X$, i.e., there exists P such that for $p \geq P$ then $L_p''(\bar{x}, l_p)(h, h) > 0$ for all $h \in X$.

6.2 Example 2

The following is an example of the case where both X and Y are infinite-dimensional Hilbert spaces (Example 2.2 found in [5]).

Let $X = L_2[0, 1], Y = L_2[0, 1] \times L_2[0, 1]$, and consider the programming problem:

$$P_2 : \min\{f(x) \mid g(x) \leq (1, 1)\}$$

where $f : X \to \mathbb{R}$ and $g : X \to Y$ are defined by

$$f(x) = -2 \int_0^{1/2} (x(t) + x(t)^2)dt + \int_{1/2}^{1} x(t)^2 dt$$
$$g(x)(t) \overset{a.e.}{=} (-x(t), x(t))$$

with

$$M = \{x \in L_2[0,1] | x(t) \overset{a.e.}{\in} [-1,1]\}$$

and the optimal $\bar{x}$:

$$\bar{x}(t) = \begin{cases} 1 & t \in [0,1/2] \\ 0 & t \in (1/2,1] \end{cases}$$

with the correspondent Lagrange multiplier $\bar{l} = (\bar{l}_1, \bar{l}_2)$ where

$$\begin{aligned} \bar{l}_1 &\overset{a.e.}{=} 0 & t &\in [0,1] \\ \bar{l}_2 &\overset{a.e.}{=} 6 & t &\in [0,1/2] \\ \bar{l}_2 &\overset{a.e.}{=} 0 & t &\in (1/2,1]. \end{aligned}$$

Clearly, there exist $x \in M$ for which $f(x) \not> 0$. Then, we need to apply the exponential transformation so the p-power formulation can be defined, which leads us to our first equivalent problem

$$P_2^{eq} : \min\{e^{f(x)} \mid (e^{g_1(x)}, e^{g_2(x)}) \overset{a.e.}{\leq} (e,e)\}.$$

Now we can define the p-power formulation as

$$P_2^p : \min\{(e^{f(x)})^p \mid ((e^{g_1(x)})^p, (e^{g_2(x)})^p) \overset{a.e.}{\leq} (e^p, e^p)\}$$

with its associated p-power Lagrangian function given by

$$L_p(x,l) = (e^{f(x)})^p + l((e^{g_1(x)})^p - e^p, (e^{g_2(x)})^p - e^p).$$

We now calculate the first derivative

$$L_p^{'}(x,l)h = \frac{d}{d\alpha}[(e^{f(x+\alpha h)})^p]\big|_{\alpha=0} + \frac{d}{d\alpha}[\langle l_1, (e^{g_1(x+\alpha h)})^p\rangle]\big|_{\alpha=0} + \frac{d}{d\alpha}[\langle l_2, (e^{g_2(x+\alpha h)})^p\rangle]\big|_{\alpha=0}$$

$$L_p^{'}(x,l)h = p(e^{f(x)})^p[-2\int_0^{1/2}(h+2xh)dt + 2\int_{1/2}^{1} xhdt] - p\int_0^1 l_1(e^{-x})^p hdt + p\int_0^1 l_2(e^x)^p hdt.$$

With the previous calculations done, we need to find the optimal Lagrangian multiplier for the p-power formulation. Since we already know $\bar{x}$, we have the formula

$$L_p^{'}(\bar{x},l)h = p[-6e^{-2p}\int_0^{1/2} hdt - e^{-p}\int_0^{1/2} l_1 hdt - \int_{1/2}^{1} l_1 hdt + e^p\int_0^{1/2} l_2 hdt + \int_{1/2}^{1} l_2 hdt] = 0.$$

We can easily check that

$$\begin{array}{llcl} \bar{l}_1(t) \overset{a.e.}{=} 0 & t & \in & [0,1] \\ \bar{l}_2(t) \overset{a.e.}{=} 6e^{-3p} & t & \in & [0,1/2] \\ \bar{l}_2(t) \overset{a.e.}{=} 0 & t & \in & (1/2,1]. \end{array}$$

Then, calculating the second Fréchet derivative

$$L_p^{''}(x,l)(h,h) = \frac{d}{d\alpha}L_p^{'}(x+\alpha h,l)h\big|_{\alpha=0}$$

we get

$$\begin{aligned} L_p^{''}(x,l)(h,h) \;&=\; p^2(e^{f(x)})^p[-2\int_0^{1/2}(h+2xh)dt + 2\int_{1/2}^{1} xhdt]^2 \\ &+p(e^{f(x)})^p[-2\int_0^{1/2} 2h^2dt + \int_{1/2}^{1} 2h^2dt] \\ &+p^2\int_0^1 l_1(e^{-x})^p h^2dt + p^2\int_0^1 l_2(e^x)^p h^2dt. \end{aligned}$$

Evaluated at $\bar{x}$ and $\bar{l}$ we get the second Fréchet derivative at the optimal point

$$L_p''(\bar{x},\bar{l})(h,h) = 36p^2e^{-2p}[\int_0^{1/2} hdt]^2 + pe^{-2p}[-4\int_0^{1/2} h^2dt + 2\int_{1/2}^{1} h^2dt] + 6p^2e^{-2p}\int_0^{1/2} h^2dt$$

and we can see that in order for the second Fréchet derivative to be positive definite, we need

$$pe^{-2p}[-4\int_0^{1/2} h^2dt + 6p\int_0^{1/2} h^2dt] > 0.$$

Then,

$$(6p-4)\int_0^{1/2} h^2dt > 0$$

and such inequality holds for $p > 2/3$. So let $p = 1$, then the equivalent problem for P_2 is finally stated as

$$P_e : \min\{e^{f(x)(t)} \mid (e^{g_1(x)(t)}, e^{g_2(x)(t)}) \overset{a.e.}{\leq} (e,e)\}$$

and the second Fréchet derivative in this case

$$L_p''(\bar{x},\bar{l})(h,h) = 36e^{-2}[\int_0^{1/2} hdt]^2 + 2e^{-2}\int_0^{1} h^2dt > 0 \qquad \forall h \neq 0 \in L_2[0,1] \tag{6.1}$$

is positive definite since the first term is nonnegative and the second term is positive.