Chapter 5

Local Convexification on Hilbert Spaces

Now that we are familiar with the local convexification of the Lagrangian function on infinite-dimensional spaces, in this chapter we will follow the methodology presented by Li and Sun in [9] to show that the p-power formulation also locally convexifies the Lagrangian on Hilbert spaces.

We will consider the nonlinear programming problem:

$$\min \quad g_0(x) \tag{5.1a}$$

$$s.t. \quad g_i(x) \le b_i \qquad i = 1, 2, ..., m \tag{5.1b}$$

$$x \in X \tag{5.1c}$$

where $g_i : X \longrightarrow \mathbb{R}$, $g_i(x) > 0$, $i = 0, ..., m$, are twice continuously differentiable functionals over a Hilbert Space X. We assume that $b_i > 0, i = 1, ..., m$.

We also need to impose a regularity condition. For this purpose, we will use the *range condition* ($g'(\bar{x})$ is surjective) under which the Karush-Kuhn-Tucker theorem was proven by Kurcyusz in [7]. Here, $g(x) = (g_1(x), ..., g_m(x))$.

Let $\bar{x}$ be a local optimal solution of (5.1) satisfying the second-order sufficient condition and let $\bar{x}$ be regular. From the Karush-Kuhn-Tucker theorem we know there exists a Lagrangian multiplier vector $l \in \mathbb{R}^m$, $l \geq 0$ such that

$$L'(\bar{x}, l)(h) = g_0'(\bar{x})h + \sum_{i=1}^{m} l_i g_i'(\bar{x})h = 0 \tag{5.2a}$$

$$\text{and} \quad \sum_{i=1}^{m} l_i(g_i(\bar{x}) - b_i) = 0, \qquad \forall h \in X. \tag{5.2b}$$

We define

$$\begin{aligned} I(\bar{x}) &= \{i \in \{1, ..., m\} |\ g_i(\bar{x}) = b_i\} \\ J(\bar{x}) &= \{i \in \{1, ..., m\} |\ l_i > 0\} \end{aligned}$$

which will be used further ahead. We will also need the second Fréchet derivative of the Lagrangian, which is

$$L''(x, l)(h, h) = g_0''(x)(h, h) + \sum_{i=1}^{m} l_i g_i''(x)(h, h). \tag{5.3}$$

If we apply the power p to (5.1), with $p \geq 1$, we get our p-power formulation as follows:

$$\min \quad [g_0(x)]^p \tag{5.4a}$$

$$s.t. \quad [g_i(x)]^p \leq b_i^p \qquad i = 1, 2, ..., m \tag{5.4b}$$

$$x \in X. \tag{5.4c}$$

Lemma 5.1 *If $\bar{x}$ is a local minimum of (5.1) then it is a local minimum of (5.4).*

Proof. Let $\bar{x}$ be a local minimum for (5.1), then $g_0(\bar{x}) < g_0(x), \forall x \in B(\bar{x}, \varepsilon)$. We can write

$$[g_0(x)]^p = e^{p \ln g_0(x)}$$

and since $p > 0$ and both $\ln(x)$ and e^x are increasing functions, we have

$$e^{p \ln g_0(\bar{x})} < e^{p \ln g_0(x)} \qquad \forall x \in B(\bar{x}, \varepsilon).$$

Also, $g_i(x) \leq b_i \iff e^{p \ln g_i(x)} \leq e^{p \ln b_i}$. Then, both programming problems are equivalent and $\bar{x}$ is a local minimum for (5.4).■

Lemma 5.2 *If $\bar{x}$ is a regular point of (5.1) then it is a regular point of (5.4).*

Proof. Let $\bar{x}$ be a regular point of (5.1), then we know that $g'(\bar{x})h$ is surjective. For (5.4), it is necessary that $p[g_i(\bar{x})]^{p-1}g_i'(\bar{x})h$ be surjective for $i = 1, ..., m$, and since $p[g_i(\bar{x})]^{p-1} > 0$ given that $p > 0$ and $g_i(\bar{x}) > 0$, we can conclude that $p[g_i(\bar{x})]^{p-1}g_i'(\bar{x})h$ for $i = 1, ..., m$ is surjective as well. ■

Now, we consider the p-power Lagrangian functional associated with (5.4):

$$L_p(x, l_p) = [g_0(x)]^p + \sum_{i=1}^{m} l_{pi}([g_i(x)]^p - b_i^p). \tag{5.5}$$

Given lemma 5.1 and lemma 5.2, $\bar{x}$ is a local minimum and a regular point, then from (3.5) there exists $l_p \in \mathbb{R}^m$, $l_p \geq 0$, such that

$$L_p'(\bar{x}, l_p)(h) = p\{[g_0(\bar{x})]^{p-1}g_0'(\bar{x})h + \sum_{i=1}^{m} l_{pi}[g_i(\bar{x})]^{p-1}g_i'(\bar{x})h\} = 0 \tag{5.6a}$$

$$\text{and} \quad \sum_{i=1}^{m} l_{pi}([g_i(\bar{x})]^p - b_i^p) = 0, \qquad \forall h \in X. \tag{5.6b}$$

From (5.2) and (5.6) we have that the optimal multiplier vector associated with x in the p-power Lagrangian is given by

$$l_{pi} = \begin{cases} \frac{[g_0(\bar{x})]^{p-1}}{b_i^{p-1}} l_i & i \in J(\bar{x}) \\ 0 & otherwise \end{cases} \tag{5.7}$$

It is our concern now that the second Fréchet derivative of the p-power Lagrangian be positive definite. To this effect, we present the following lemma which will be used in the proof of the theorem that will guarantee the existence of a sufficiently large p, for which the p-power Lagrangian is convex.

Lemma 5.3 *Let X be a Hilbert space. Then, the closed unit sphere is compact in X.*

Proof. By the theorem of Aloaglu we know that the closed unit sphere is weak* compact in X. It follows that given an arbitrary sequence $\{x_i^*\}$ in the closed unit sphere, there will be a subsequence $\{x_{i_n}^*\}$ converging to an element x^* in the sphere. Since the dual of the Hilbert spaces are the spaces themselves, we can easily see that for every sequence $\{x_i\}$ in a Hilbert space, there will be a subsequence $\{x_{i_n}\}$ converging to an element x in the Hilbert space itself.

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Theorem 5.1 *Let $\bar{x}$ be a local optimal solution of (5.1). Assume that $J(\bar{x}) \neq \emptyset$ and that x is a regular point that satisfies the second-order sufficiency condition (theorem 3.6). Then there exists a $q > 0$ such that the second Fréchet derivative of the p-power Lagrangian is positive definite when $p > q$.*

Proof. Let $l_0 = 1$, $\bar{J}(\bar{x}) = J(\bar{x}) \cup \{0\}$, then

$$\begin{aligned} L_p''(\bar{x}, l_p)(h,h) &= \sum_{i=0}^{m} l_i p[g_i(\bar{x})]^{p-2}[(p-1)(g_i'(\bar{x})h)^2 + g_i(\bar{x})g_i''(\bar{x})(h,h)] \\ &= p[g_0(\bar{x})]^{p-1}\{\sum_{i\in\bar{J}(\bar{x})} l_{pi}[(p-1)\frac{{b_i}^{p-2}}{[g_0(\bar{x})]^{p-1}}(g_i'(x)h)^2 + \frac{{b_i}^{p-1}}{[g_0(\bar{x})^{p-1}]}g_i''(\bar{x})(h,h)]\} \\ &= p[g_0(\bar{x})]^{p-1}\{\sum_{i\in\bar{J}(\bar{x})} [(p-1)\frac{l_i}{g_i(\bar{x})}(g_i'(x)h)^2 + l_i g_i''(\bar{x})(h,h)]\} \end{aligned}$$

$$= p[g_0(\bar{x})]^{p-1}\{L''(\bar{x},l)(h,h) + (p-1)\sum_{i\in\bar{J}(\bar{x})} u_i(g_i'(x)h)^2\} \tag{5.8}$$

where $u_i = \frac{l_i}{g_i(\bar{x})}$, and $u_i > 0$ for all $i \in \bar{J}(\bar{x})$.

Since we know $\overline{x}$ is regular and satisfies the second-order sufficiency condition, from theorem 3.6 we have that there exist $\delta > 0$ and $\beta > 0$ such that

$$L''(\overline{x}, l)(h,h) \geq \delta \|h\|^2 \tag{5.9}$$

$$\text{for all} \quad h \in L(M,\overline{x}) \cap \left\{h \in X \mid l \cdot g'(\overline{x})h \leq \beta \|h\|\right\} \tag{5.10}$$

i.e., for all $h \in X$ such that there exist $k \in K$ and $\lambda \in \mathbb{R}$ with

$$g'(\overline{x})h = k + \lambda\left(g(\overline{x}) - b\right) \text{ and } l \cdot g'(\overline{x})h \leq \beta \left\| h \right\|. \tag{5.11}$$

Let

$$T(\overline{x}) := \left\{h \in X \mid g_i'(\overline{x})h = 0, i \in J(\overline{x})\right\}.$$

We will consider two cases.

Case 1. $T(\overline{x}) = \{0\}$.

We will denote the unit sphere in X as $S := \{h \in X \mid \|h\| = 1\}$.Since X is a Hilbert space, from lemma 5.3 we have that S is compact. Then we can define

$$\eta := \min_{h \in S} L''(\overline{x}, l)(h, h)$$

which we know its existance from the Weierstrass' theorem. We will also define

$$\tau := \min_{h \in S} \sum_{j \in \overline{J}(\overline{x})} u_j \left(g_j'(\overline{x})h\right)^2 > 0$$

given that $S \cap T(\overline{x}) = \emptyset$ implies $\sum_{j \in \overline{J}(\overline{x})} u_j \left(g_j'(\overline{x})h\right)^2 > 0$. Furthermore, $\sum_{j \in \overline{J}(\overline{x})} u_j \left(g_j'(\overline{x})h\right)^2$ is continuous over S and S is compact. Then, we have

$$L_p''(\overline{x}, l)(h, h) \geq p\left[g_0(\overline{x})\right]^{p-1}\left\{\eta + (p-1)\tau\right\} \qquad \forall h \in S.$$

Then, if $p > q_1 = \max\left\{1, 1 - \frac{\eta}{\tau}\right\}$, we get

$$L_p''(\overline{x}, l)(h, h) > 0 \quad \forall h \in S$$

and hence

$$L_p''(\overline{x}, l)(h, h) > 0 \quad \forall h \in X.$$

Case 2. $T(\overline{x}) \neq \{0\}$.

Note that from (5.10) we know there exist $\lambda \in \mathbb{R}$ and $k_i \leq 0$, $i = 1, \ldots, m$ such that

$$g_i'(\overline{x})h = k_i + \lambda\left(g_i\left(\overline{x}\right) - b_i\right) = \begin{cases} k_i, & \text{si } i \in I(\overline{x})\,(\supset J(\overline{x})) \\ k_i + \lambda\left(g_i\left(\overline{x}\right) - b_i\right), & \text{si } i \notin I(\overline{x}). \end{cases} \tag{5.12}$$

We can see that $T(\overline{x}) \subset L(M, \overline{x})$ because we only need to take $k_i = 0$ in (5.12) for $i \in J(\overline{x})$. Then by (5.9) and Weierstrass' theorem, we have that

$$\epsilon = \min_{h \in T(\overline{x}) \cap S} L''(\overline{x}, l)(h, h) \geq \delta \left\|h\right\|^2 = \delta > 0 \tag{5.13}$$

Note that in (5.13) $T(\overline{x})$ is closed and S is compact. Then $T(\overline{x}) \cap S$ is a closed set contained in the compact set S,which implies that $T(\overline{x}) \cap S$ is compact.

For $\theta_0 > 0$, denote

$$S_T(\overline{x}, \theta_0) := \left\{h \in S \mid \left|g_i'(\overline{x})h\right| \leq \theta_0, \forall i \in J(\overline{x})\right\}$$

$S_T(\overline{x}, \theta_0)$ is a closed set contained in the compact set S, then $S_T(\overline{x}, \theta_0)$ is compact and $T(\overline{x}) \cap S \subset S_T(\overline{x}, \theta_0)$. Then, by the continuity of $L''(\overline{x}, l)(h, h)$ and (5.13), we have that there exists $\theta_0 > 0$ such that

$$L''(\overline{x}, l)(h, h) \geq \frac{\epsilon}{2} \qquad \forall h \in S_T(\overline{x}, \theta_0).$$

Hence $\forall h \in S_T(\overline{x}, \theta_0)$ and $\forall p \geq 1$:

$$L_p''(\overline{x}, l)(h, h) \geq \frac{\epsilon}{2} p\left[f\left(\overline{x}\right)\right]^{p-1} > 0. \tag{5.14}$$

On the other hand, if $h \in S \setminus S_T(\overline{x}, \theta_0)$, then there exists $i_0 \in J(\overline{x})$ with

$$\left|g_{i_0}'(\overline{x})h\right| > \theta_0.$$

Let

$$c = \min \left\{ \left| g_i'(\overline{x})h \right| \mid \left| g_i'(\overline{x})h \right| \geq \theta_0 \right\}.$$

Note that $\theta_0 > 0 \Rightarrow c > 0$, and denote

$$\overline{u} = \min_{i \in J(\overline{x})} u_j > 0$$

given that $u_i > 0$ for all $i \in J(\overline{x})$.

Now, since $S \setminus S_T(\overline{x}, \theta_0) \subset S \Rightarrow$

$$\min_{h \in S \setminus S_T(\overline{x},\theta_0)} L''(\overline{x}, l)(h, h) \geq \min_{h \in S} L''(\overline{x}, l)(h, h) = \eta$$

then,

$$\begin{aligned} L_p''(\overline{x}, l)(h, h) &\geq p\left[f\left(\overline{x}\right)\right]^{p-1}\left(\eta + (p-1)u_{i_0}\left(g_{i_0}'\left(\overline{x}\right)h\right)^2\right) \\ &\geq p\left[f\left(\overline{x}\right)\right]^{p-1}\left(\eta + (p-1)\overline{u}c^2\right). \end{aligned}$$

Here, we can see that if $p > q_2 = \max\left\{1, 1 - \frac{\eta}{\overline{u}c^2}\right\}$, then

$$L_p''(\overline{x}, l)(h, h) > 0 \qquad \forall h \in S \setminus S_T(\overline{x}, \theta_0). \tag{5.15}$$

Finally, (5.14) together with (5.15) imply that if $p > q = \max\{q_1, q_2\}$,

$$L_p''(\overline{x}, l)(h, h) > 0 \qquad \forall h \in S$$

and hence

$$L_p''(\overline{x}, l)(h, h) > 0 \qquad \forall h \in X. \quad \blacksquare$$