Chapter 4

Local Convexification on Finite Dimensional Spaces

In this chapter we will present results of D. Li and X.L. Sun's investigation found in [9].

We will consider a constrained noncovex optimization problem of the following form:

$$\min \quad f_0(x) \tag{4.1a}$$

$$s.t. \quad f_j(x) \le b_j \qquad j = 1, 2, ..., m \tag{4.1b}$$

$$x \in X \tag{4.1c}$$

where $f_j : \mathbb{R}^n \longmapsto \mathbb{R}$, $j = 0, 1, ..., m$, are twice continuously differentiable functions and X is a nonempty closed set in $\mathbb{R}^n$.

It is also assumed that $f_j(x)$, $j = 0, ..., m$, are positive over X and that b_j, $j = 1, ..., m$, are positive. This assumption does not impose a strict restriction on problem (4.1) since we can always apply a suitable equivalent transformation on (4.1) if necessary (e.g. the exponential transformation).

Let x^* be a local optimal solution of problem (4.1). The Lagrangian function associated to problem (4.1) is given by:

$$L(x,\lambda) = f_0(x) + \sum_{j=1}^{m} \lambda_j [f_j(x) - b_j] \qquad \lambda \geq 0. \tag{4.2}$$

A fundamental condition to apply local duality theory at x^* is that the Hessian of Lagrangian $L(x^*, \lambda^*)$ be positive definite, where λ^* is the Lagrangian multiplier vector corresponding to x^*. However, many nonconvex problems do not satisfy this local convexity condition at their local optimal solutions. Hence, applying dual methods to this kind of problems will fail to solve the primary problem.

In Li and Sun's paper, they show that the local convexity condition in the local duality theorem can be achieved at a local optimal solution of (4.1) after adopting some suitable convexification transformation on problem (4.1). They prove that under certain regular and second-order sufficiency conditions, the Hessian of the p-power Lagrangian is positive definite at x^* for sufficiently large p.

Then, we will show the convexification theorems for the pth power Lagrangian given in [9].

4.1 Local Convexity of the p-Power Lagrangian

Let x^* be a local optimal solution of (4.1). Assume that x^* is an interior point of X and x^* is a regular point of the constraints in (4.1), i.e., $\nabla f_j(x), j \in I(x)$ are linearly independent, where $I(x) = \{j | f_j(x) = b_j, j = 1, ..., m\}$.

Also, suppose x^* satisfies the second-order sufficiency condition. Then, there exists a Lagrangian multiplier vector $\lambda^* \geq 0$ such that

$$\nabla f_0(x^*) + \sum_{j=1}^{m} \lambda_j^* \nabla f_j(x^*) = 0 \tag{4.3}$$

$$\lambda_j^* [f_j(x^*) - b_j] = 0 \qquad j = 1, ..., m \tag{4.4}$$

and the Hessian matrix

$$\nabla_x^2 L(x^*, \lambda^*) = \nabla_x^2 f_0(x^*) + \sum_{j=1}^{m} \lambda_j^* \nabla_x^2 f_j(x^*)$$

is positive definite on the tangent subspace

$$M(x^*) = \{d \in \mathbb{R}^n | d^T \nabla f_j(x^*) = 0, \forall j \in J(x^*)\}$$

where $J(x^*) = \{j | \lambda_j^* > 0, j = 1, ..., m\}$. Thus, we have

$$d^T \nabla_x^2 L(x^*, \lambda^*) d > 0 \qquad \forall d \in M(x^*) \qquad d \neq 0. \tag{4.5}$$

The goal is to find an equivalent programming problem such that $\nabla_x^2 L(x^*, \lambda^*)$ is positive definite on all $\mathbb{R}^n$, so that the Lagrangian function is convex on a neighborhood of x^*. To this purpose, the p-power formulation of (4.1) is defined as:

$$\min \quad [f_0(x)]^p \tag{4.6a}$$

$$s.t. \quad [f_j(x)]^p \leq b_j^p \qquad j = 1, 2, ..., m \tag{4.6b}$$

$$x \in X \tag{4.6c}$$

with the p-power Lagrangian function associated to (4.6):

$$L_p(x, \mu) = [f_0(x)]^p + \sum_{j=1}^{m} \mu_j \{[f_j(x)]^p - b_j^p\} \tag{4.7}$$

for $p > 0$ and $\mu \geq 0$, then from (4.3) and (4.4) the optimal multipliers associated with x^* in the p-power Lagrangian (4.7) are given by

$$(\mu_p^*)_j = \begin{cases} [f_0(x^*)]^{p-1} / b_j^{p-1} \lambda_j^* & j \in J(x^*) \\ 0 & otherwise \end{cases} \tag{4.8}$$

since

$$\begin{aligned}
\nabla_x L_p(x,\mu)\big|_{x=x^*} &= p[f_0(x^*)]^{p-1}\nabla f_0(x^*) + \sum_{j=1}^{m} \mu_j^* \{p[f_j(x^*)]^{p-1}\nabla f_j(x^*)\} \\
&= p\{[f_0(x^*)]^{p-1}\nabla f_0(x^*) + \sum_{j=1}^{m} \mu_j^* [f_j(x^*)]^{p-1}\nabla f_j(x^*)\} \\
&= \nabla f_0(x^*) + \sum_{j=1}^{m} \mu_j^* \frac{[f_j(x^*)]^{p-1}}{[f_0(x^*)]^{p-1}} \nabla f_j(x^*)
\end{aligned}$$

from (4.3) we can see that

$$\lambda_j^* = \frac{[f_j(x^*)]^{p-1}}{[f_0(x^*)]^{p-1}}\mu_j^*$$

so if $j \in J(x^*)$, we have $f_j(x^*) = b_j$. Therefore,

$$\lambda_j^* = \frac{b_j^{p-1}}{[f_0(x^*)]^{p-1}}\mu_j^* \tag{4.9}$$

and in the case where $j \notin J(x^*)$, we have $\lambda_j^* = 0$, which brings us to $\mu_j^* = 0$.

Now that we have presented the p-power formulation we will introduce the first theorem shown and proven by Li and Sun in [9].

Theorem 4.1 *Let x^* be a local optimal solution of (4.1). Assume that $J(x^*) \neq \varnothing$, x^* is a regular point and x^* satisfies the second-order sufficiency condition. Then there exists a $q_1 > 0$ such that the Hessian matrix $\nabla_x^2 L_p(x^*, \mu_p^*)$ of the p-power Lagrangian is positive definite when $p > q_1$.*

Define the dual function for (4.6) as

$$\phi_p(\mu) = \min_{x\in X} L_p(x,\mu)$$

where minimization is to be understood as taken locally with respect to x near x^*.

Since $L_p(x,\mu)$ is the Lagrangian associated with problem (4.6), which is equivalent to (4.1), and since the above theorem guarantees the local convexity condition for $L_p(x,\mu)$ required by the local dual theorem in [11] (Section 13.1), the following corollary is obtained.

Corollary 4.1 *Let* x^* *be a local solution of (4.1) with optimal value* r *and Lagrangian multiplier* λ^*. *Under the assumptions in Theorem 3.1, there exists a* $q_1 > 0$ *such that, when* $p > q_1$ *the dual problem* $\max_{\mu \geq 0} \phi_p(\mu)$ *has a local solution* μ_p^* *defined by (4.8) with the optimal value* r^p *and* $L_p(x^*, \mu_p^*) = \phi_p(\mu_p^*)$.