Chapter 2

Basic Concepts

In this chapter we will review some basic ideas and concepts that can be found in [10] and are used in the development of this work.

First, we will mention the definition of Cauchy sequences in order to accurately define Banach spaces. Then, we will present the definition of Hilbert spaces and list some of their properties that will be useful in succeeding chapters.

2.1 Cauchy sequences

Definition 2.1 *A sequence* $\{x_n\}$ *in a normed space is said to be a* Cauchy sequence *if* $\|x_n - x_m\| \to 0$ *as* $n, m \to \infty$*; i.e., given* $\varepsilon > 0$*, there is an integer* N *such that* $\|x_n - x_m\| < \varepsilon$ *for all* $n, m > N$*.*

Remark. In a normed space, every convergent sequence is a Cauchy sequence.

Definition 2.2 *A normed linear vector space* X *is* complete *if every Cauchy sequence from* X *has a limit in* X*.*

2.2 Banach spaces

Definition 2.3 *A complete normed linear vector space is called a Banach space.*

Theorem 2.1 *In a Banach space a subset is complete if and only if it is closed.*

2.3 Hilbert spaces

Definition 2.4 *A* pre-Hilbert space *is a linear vector space X together with an inner product defined on $X \times X$. Corresponding to each pair of vectors x and $y \in X$ the inner product $\langle x, y\rangle$ of x and y is a scalar.*

The inner product satisfies the following axioms:

1.$\langle x, y\rangle = \langle y, x\rangle$

2.$\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

3.$\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$ for $\lambda \in \mathbb{R}$.

4.$\langle x, y\rangle \geq 0$ and $\langle x, x\rangle = 0$ iff $x = \theta$.

Remark. On a pre-Hilbert space X the function $\|x\| = \sqrt{\langle x, y\rangle}$ defines the norm over X.

Definition 2.5 *A complete pre-Hilbert space is called a* Hilbert space*.*

Then, a Hilbert space is a Banach space equipped with an inner product which induces the norm. The spaces $\mathbb{R}^n$, l_2 and $L_2[a, b]$ are Hilbert spaces.

Example 2.1 $\mathbb{R}^n$ *is a Hilbert space with the inner product defined as $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$, where $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$. The norm defined as $\sqrt{\langle x, y\rangle}$ is $\|x\| = (\sum_{i=1}^{n} |x_i|^2)$ which is the Euclidean norm of* $\mathbb{R}^n$*.*

Example 2.2 *The space l_2 is a Hilbert space with the inner product defined as $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$, where $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$.*

Example 2.3 *The space $L_2[a, b]$ is a Hilbert space with the inner product defined as $\langle x, y\rangle = \int_a^b f(x)g(x)dx$.*

Lemma 2.1 *In a Hilbert space the statement $\langle x, y\rangle = 0$ for all y implies that $x = \theta$.*

Lemma 2.2 *(Continuity of the Inner Product) Suppose that $x_n \to x$ and $y_n \to y$ in a Hilbert space. Then $\langle x_n, y_n \rangle \to \langle x, y \rangle$.*

2.4 Dual spaces

Definition 2.6 *Let X be a normed linear vector space. The space of all bounded linear functionals on X is called the* normed dual of X *and is denoted X^*. The norm of an element $f \in X^*$ is $\|f\| = \sup_{\|x\| \leq 1} |f(x)|$.*

Theorem 2.2 *X^* is a Banach space.*

The Dual of Hilbert Space. On Hilbert spaces, bounded linear functionals are generated by elements of the space itself. Consequently, the dual of the Hilbert spaces $\mathbb{R}^n$, l_2 and L_2 are the spaces themselves.

2.5 Convergence and Compactness

Definition 2.7 *A set K in a normed space X is said to be* compact *if, given an arbitrary sequence $\{x_i\}$ in K, there is a subsequence $\{x_{i_n}\}$ converging to an element $x \in K$.*

In finite dimensions, compactness is equivalent to being closed and bounded, but this is not true in general normed spaces. Note, however, that a compact set K must be complete since any Cauchy sequence from K must have a limit in K.

Theorem 2.3 *(Weierstrass) An upper semicontinuous functional on a compact subset K of a normed linear space X achieves a maximum on K.*

Definition 2.8 *In a normed linear space an infinite sequence of vectors $\{x_n\}$ is said to* converge *to a vector x if the sequence $\{\|x - x_n\|\}$ of real numbers converges to zero. In this case, we write $x_n \to x$. This definition of convergence is sometimes referred to as* strong convergence.

Definition 2.9 *A sequence in a normed linear vector space X is said to* converge weakly *to $x \in X$ if for every $x^* \in X^*$ we have $\langle x_n, x^* \rangle \to \langle x, x^* \rangle$.*

Definition 2.10 *A sequence $\{x_n^*\}$ in X^* is said to* converge weak-star *(or weak*) to the element x^* if for every $x \in X$, $\langle x_n, x_n^* \rangle \to \langle x, x^* \rangle$.*

Definition 2.11 *A set $K \subset X^*$ is said to be* weak* compact *if every infinite sequence from K contains a weak* convergent subsequence.*

Theorem 2.4 *(Alaoglu) Let X be a real normed linear space. The closed unit sphere in X^* is weak* compact.*