Chapter 7
Results and Future Work

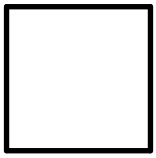
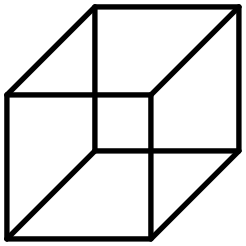
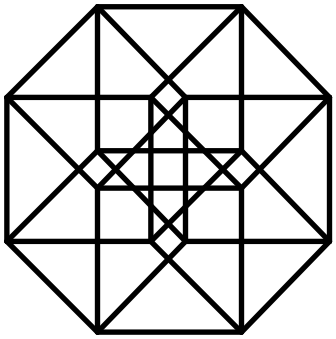
7.1 Results: Resume of Studied Analogies

7.1.1 Analogies Between The Parallelotopes In the 1D, 2D, 3D and 4D space

In chapter 2 were studied the properties of the segment, the square, the cube and the hypercube (see **Table 7.1**):

- Segment: 1D Parallelotope (C_1), boundary composed by 2 vertices.
- Square: 2D Parallelotope (C_2), boundary composed by 4 vertices and 4 edges.
- Cube: 3D Parallelotope (C_3), boundary composed by 8 vertices, 12 edges and 6 faces.
- Hypercube: 4D Parallelotope (C_4), boundary composed by 16 vertices, 32 edges, 24 faces and 8 volumes.

TABLE 7.1
Analogies between the nD Parallelotopes (1D, 2D, 3D and 4D; own elaboration)

Segment	Square	Cube	Hypercube
			

7.1.2 Analogies Between The Main Rotations In The 2D, 3D and 4D space

In chapter 3 were presented the possible main rotations in the 2D, 3D and 4D space:

- 2D space: Rotation around a point, 1 possible main rotation around the origin.
- 3D space: Rotation around an axis, 3 possible main rotations around main axis (X, Y, and Z).
- 4D space: Rotation around a plane, 6 possible main rotations around main planes (XY, YZ, XZ, XW, YW, and ZW) [Hollasch, 91].

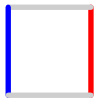
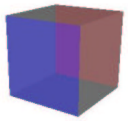
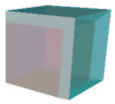
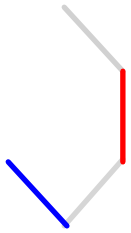
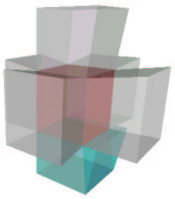
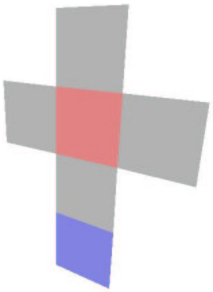
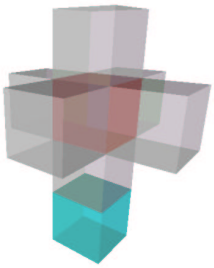
7.1.3 Analogies Between The Unravellings Of The Parallelotopes In the 2D, 3D and 4D space

In chapter 4 were studied the properties of the unravellings for the parallelotopes in 2D, 3D and 4D space (see **Table 7.2**):

- Square: A central segment surrounded by other two through a vertex adjacency with each one; a satellite segment adjacent to any other of the segments except the central. Completely immersed in a 1D space (a straight line).
- Cube: A central face surrounded by other four through an edge adjacency with each one; a satellite face adjacent to any other of the faces except the central. Completely immersed in a 2D space (a plane).
- Hypercube: A central volume surrounded by other six through a face adjacency with each one; a satellite volume adjacent to any other of the volumes except the central. Completely immersed in a 3D space (a hyperplane).

TABLE 7.2

Analogies between the unravellings of the square, the cube and the hypercube (the central cell is shown in red and the satellite cell is shown in blue; own elaboration).

	2D Space Square	3D Space Cube	4D Space Hypercube
Parallelotope			
Unraveling process			
Unravellings			

7.1.4 Analogies Between n-2 Dimensional Elements in 2D, 3D and 4D-OPP's

We know since [Aguilera, 98] that there are two types of vertices in 2D-OPP's:

- The manifold vertex with two perpendicular incident edges.
- The non-manifold vertex with four incident edges.

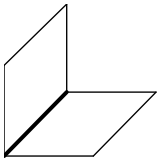
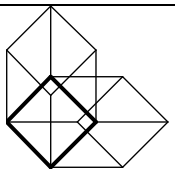
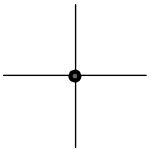
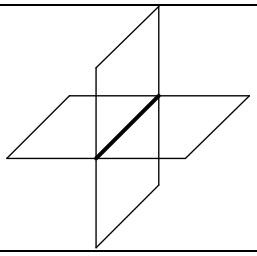
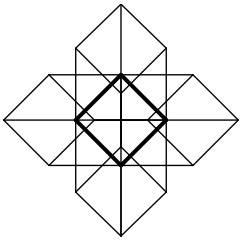
Also, [Aguilera, 98] describes two types of edges in 3D-OPP's:

- The manifold edge with two perpendicular incident faces.
- The non-manifold face with four incident faces.

Finally, we know from section 5.3.3 that there are identified two types of faces in 4D-OPP's (**Table 7.3**):

- The manifold face with two perpendicular incident volumes.
- The non-manifold face with four incident volumes.

TABLE 7.3
Analogies between the n-2 dimensional elements in nD-OPP's (2D, 3D and 4D; own elaboration).

	Vertex in 2D-OPP's	**Edges in 3D-OPP's**	**Faces in 4D-OPP's**
Analogous Manifold			
Analogous Non-manifold			

7.1.5 Analogies Between n-3 Dimensional Elements in 3D and 4D-OPP's

We know since [Aguilera, 98] that there are eight types of vertices in 3D-OPP's:

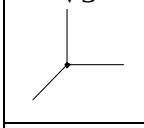
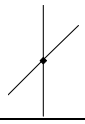
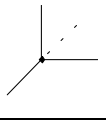
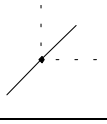
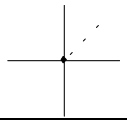
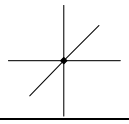
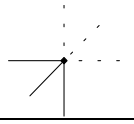
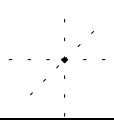
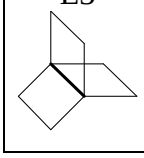
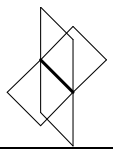
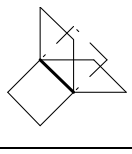
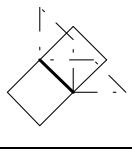
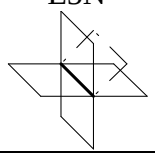
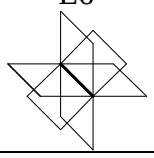
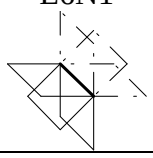
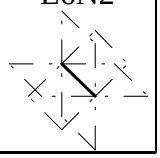
- The manifold and extreme vertex V3.
- The manifold and non-extreme vertex V4.
- The non-manifold and extreme vertex V4N1.

- The non-manifold and non-extreme vertex V4N2.
- The non-manifold and non-extreme vertex V5N.
- The manifold or non-manifold (depending the case), and non-extreme vertex V6.
- The non-manifold and extreme vertex V6N1.
- The non-manifold and non-extreme vertex V6N2.

From sections 5.4.4 and 6.3 we have identified eight types of edges in 4D-OPP's (**Table 7.4**):

- The manifold and extreme edge E3.
- The manifold and non-extreme edge E4.
- The non-manifold and extreme edge E4N1.
- The non-manifold and non-extreme edge E4N2.
- The non-manifold and non-extreme edge E5N.
- The manifold or non-manifold (depending the case), and non-extreme edge E6.
- The non-manifold and extreme edge E6N1.
- The non-manifold and non-extreme edge E6N2.

TABLE 7.4
Analogies between the n-3 dimensional elements in nD-OPP's
(3D first row, and 4D second row; own elaboration)

V3	V4	V4N1	V4N2	V5N	V6	V6N1	V6N2
							
E3	E4	E4N1	E4N2	E5N	E6	E6N1	E6N2
							

7.2 Using The Studied Analogies And Their Proposed Generalizations To Predict Some Properties Of The 5D Space

7.2.1 The Properties Of The 5D Hypercube

Applying the formula presented by [Coxeter, 63] (Chapter 2):

$$Q(n,k) = \frac{2^n \cdot C(n,k)}{2^k} = 2^{n-k} \cdot C(n,k)$$

We have that the 5D hypercube's boundary is composed by:

- 32 (C_0's) Vertices (k = 0, n = 5)
- 80 (C_1's) Edges (k = 1, n = 5)
- 80 (C_2's) Faces (k = 2, n = 5)
- 40 (C_3's) Volumes (k = 3, n = 5)
- 10 (C_4's) Hypervolumes (k = 4, n = 5)

Applying the distribution for the coordinates for the nD hypercube [Coxeter, 63]:

$$C\binom{n}{0},\ C\binom{n}{1},\ \ldots,\ C\binom{n}{i},\ \ldots,\ C\binom{n}{n-1},\ C\binom{n}{n}$$

where $C\binom{n}{i} = \frac{n!}{i!(n-i)!}$ defines the number of those coordinates that have i ones and $n-i$ zeros. Using n = 5 we have the coordinates presented in **Table 7.5**.

TABLE 7.5
The 5D Hypercube's Coordinates (own elaboration).

i	Combinations	Coordinates
0	$C\binom{5}{0}=1$	(0,0,0,0,0)
1	$C\binom{5}{1}=5$	(1,0,0,0,0) (0,1,0,0,0) (0,0,1,0,0) (0,0,0,1,0) (0,0,0,0,1)
2	$C\binom{5}{2}=10$	(1,1,0,0,0) (1,0,1,0,0) (1,0,0,1,0) (1,0,0,0,1) (0,0,0,1,1) (0,0,1,0,1) (0,1,0,0,1) (0,1,1,0,0) (0,0,1,1,0) (0,1,0,1,0)
3	$C\binom{5}{3}=10$	(0,0,1,1,1) (0,1,0,1,1) (0,1,1,0,1) (0,1,1,1,0) (1,1,1,0,0) (1,1,0,1,0) (1,0,1,1,0) (1,0,0,1,1) (1,1,0,0,1) (1,0,1,0,1)
4	$C\binom{5}{4}=5$	(0,1,1,1,1) (1,0,1,1,1) (1,1,0,1,1) (1,1,1,0,1) (1,1,1,1,0)
5	$C\binom{5}{5}=1$	(1,1,1,1,1)

7.2.2 The Main Rotations In 5D Space

[Duffin, 94] presents the following general rotation matrix (Chapter 3):

$$R_{ab}(\theta) = \begin{bmatrix} & r_{ii} = 1 & i \neq a, i \neq b \\ & r_{aa} = \cos\theta & \\ & r_{bb} = \cos\theta & \\ r_{ij} & r_{ab} = -sin\theta & \\ & r_{ba} = sin\theta & \\ & r_{ij} = 0 & elsewhere \end{bmatrix}$$

Which indicates the rotation of the axis Xa in direction to axis Xb. For a nD space we have that there are $C\binom{n}{2}$ main rotation planes leaving a (n-2)D subspace fixed during the rotation. For n = 5 then we have 10 main rotation planes, which during the rotation leave a 3D subspace fixed (rotation around a volume). In **Table 7.6** we have the 5D main rotation planes (the 5D space's fifth axis is presented as T).

TABLE 7.6
The 5D Space main rotations (own elaboration).

Rotation Plane	Subspace Fixed During Rotation
XY	ZWT
YZ	XWT
XZ	YWT
XW	YZT
YW	XZY
ZW	XYT
XT	YZW
YT	XZW
ZT	XYW
WT	XYZ

7.2.3 The Unravellings Of The 5D Hypercube

In chapter 3 was presented the ***n-dimensional hyper-tesseract*** (n≥1) [Aguilera, 02b] as the result of the (n+1)-dimensional hypercube's unraveling with the following properties:

- The (n+1)-dimensional hypercube will have 2(n+1) n-dimensional cells on its boundary.
- A central cell will be static during the unraveling/raveling process.

- 2(n+1)-2 cells are adjacent to central cell. All of them will share a (n-1)-dimensional cell with central cell.
- A satellite cell won't be adjacent to central cell because their supporting hyperplanes are parallel. It will be adjacent to any of the adjacent cells (it will share a (n-1)-dimensional cell with the selected adjacent cell).

Then, for n = 4 we have the 4D hyper-tesseract as the result of the 5D hypercube's unraveling. The 4D hyper-tesseract will be composed by 10 hypervolumes, where one of them will be the central hypervolume (static), eight of them are adjacent to central hypervolume (they share a volume) and the last one will be the satellite hypervolume (it shares a volume with any of the adjacent hypervolumes). See **Figure 7.1**. The adjacent hypervolumes and the satellite hypervolume will rotate around a volume or a hyperplane during the unraveling/raveling process.

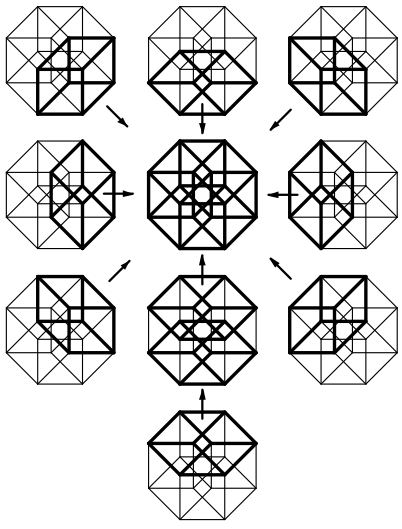


FIGURE 7.1
The possible adjacency relations between the 4D *hyper-tesseract*' s (the 5D hypercube's unravellings) central hypervolume and adjacent hypervolumes [Aguilera, 01].

7.2.4 The n-2 Dimensional Elements in 5D-OPP's

Since section 5.3.4, we have the generalized conditions to classify a Π_{n-2} as manifold or non-manifold in a nD-OPP [Aguilera, 02]:

- If two perpendicular Π_{n-1}'s are incident to a Π_{n-2} then it must be classified as **manifold**.
- If four Π_{n-1}'s are incident to a Π_{n-2} then it must be classified as **non-manifold**.

Then for 5D-OPP's we can expect two types of volumes:

- The **manifold volume** with two perpendicular and incident hypervolumes.
- The **non-manifold volume** with four incident hypervolumes.

7.2.5 The n-3 Dimensional Elements in 5D-OPP's

Volumes, faces, edges and vertices, as boundary elements for 5D polytopes, may be either manifold or non-manifold elements. We have proposed, in the last section, that a manifold volume is adjacent to exactly two hyper-volumes, and now we suggest that a manifold face is the common face (apex) of only one **5D hyper-cone of hypervolumes**. Conversely, we have suggested that a non-manifold volume is adjacent to more than two hypervolumes, and now we suggest that a non-manifold face is the common face (apex) of more than one **5D hyper-cone of hypervolumes**.

Using the general algorithm presented in section 5.4.5, we can propose the following one to classify a face, as manifold or non-manifold, in any 5D polytope:

1 Get the set of 4D Π_4's that are incident to face *A* (a Π_2).

2 From the set of Π_4's select one of them.

3 The selected Π_4 has two Π_3's that are incident to *A*, get one of them and label it as *START* and *ANOTHER*.

4 Repeat

4.1 If the number of incident Π_4's to *ANOTHER* is more than one, then *A* is a non-manifold Π_2.

4.2 The *ANOTHER* Π_3 is common to another Π_4, find it.

4.3 The Π_4 has another Π_3 that is common to *A*, find it and label it as *ANOTHER*.

4.4 Until *START* = *ANOTHER* (it has been found a 5D hyper-cone of hypervolumes).

5 If there are more Π_4's to analyze then *A* is non-manifold (there are more 5D hyper-cones of hypervolumes).

6 Otherwise, *A* is manifold (*A* is the common face of only one 5D hyper-cone of hypervolumes).

7.3 Future Work: Extreme Vertices in 4D-OPP's

7.3.1 Describing vertices in 4D-OPP's in terms of manifold and non-manifold edges, and the problem of extending the 3D Extreme Vertices' properties to the 4D space

Since section 5.4.4 we have a mechanism to classify an edge in 4D polytopes as manifold or non-manifold through its hyper-cones of volumes. Applying this mechanism, it has been found an analogy between vertices in 3D-OPP's and edges in 4D-OPP's. In section

5.4.1 were presented the eight possible vertices in 3D-OPP's identified by [Aguilera, 98]. These vertices can be classified depending on the number of manifold and non-manifold edges incident to them.

Applying the same procedure used by [Aguilera, 98], it has been found that there are 53 possible vertices in 4D-OPP's described in terms of the manifold and non-manifold edges incident to them. All these vertices are presented in **Appendix C**.

In section 6.1, the general properties for Extreme Vertices in 1D, 2D and 3D spaces were presented. We now just extend these properties to the 4D space in the following way:

- Every Extreme Vertex in a 4D-OPP has exactly 4 incident manifold edges perpendicular to each other.
- Any Extreme Vertex of a 4D-OPP, when is locally described by a set of surrounding hyper-boxes, is surrounded by an odd number of such hyper-boxes. An even number of surrounding hyper-boxes either defines a non-extreme vertex, or does not define any vertex at all.

In this way, we can resume that a vertex in a 4D-OPP ***could*** be Extreme if it has four incident manifold edges perpendicular to each other and an odd number of surrounding hyper-boxes. Unfortunately, analyzing the vertices presented in **Appendix C**, we have found that most of those vertices with an odd number of surrounding hyper-boxes don't have exactly four lineally independent and manifold incident edges. It is obvious that trying

to build extended edges with this vertices won't give us a solution, because the brink's ending vertices don't correspond to the expected results.

7.3.2 The Extreme Edges in 4D-OPP's As A Possible Solution To The Problem Of Extending The 3D Extreme Vertices' properties to the 4D space

In the previous section was presented the problem between the extended properties for Extreme Vertices in the 4D space and the identified vertices, described in terms of manifold and non-manifold edges, presented in **Appendix C**. In resume, it has been found that for an odd number of surrounding hyper-boxes, the expected vertex with exactly four lineally independent and manifold incident edges is not present if most of the expected cases.

When we change in the first extended property (presented in previous section) that should satisfy those vertices to be Extreme in the 4D space, the term "manifold edges" for "extreme edges", we have then:

- Every Extreme Vertex in a 4D-OPP has exactly 4 incident ***extreme*** edges perpendicular to each other.

Analyzing now, the possible vertices in 4D-OPP's described in terms of extreme and non-extreme edges, which are presented in **Table 6.5**, we have that one of the most interesting properties is that such vertices, with four incident and lineally independent extreme edges, have an odd number of surrounding hyper-boxes. In **Appendix D** is shown each vertex and its corresponding configurations, where this property is clearly visible.