Chapter 1
Introduction

1.1 Historical Overview

In the ancient Greek, Euclid said that "a point has no dimension at all. A line has only one dimension: length. A plane has two dimensions: length and breadth. A solid has three dimensions: length, breadth, and height. And there it stops. Nothing has four dimensions".

In *"The Republic"* (370 b.C.) is presented the Plato's allegory of the cave. [Gutiérrez, 95] resumes it as follows: in a dark cave there are some prisoners chained since they were children. They can't see the daylight, the objects nor the people from the exterior. They just see the shadows that are projected onto the bottom's cave. Outside the cave there are a road and a torch that origins these shadows. The prisoners consider the shadows as their only reality. One of the prisoners escapes and discovers the real world. He returns to the cave and try to convince the others. They don't believe him. An important aspect of the Plato's allegory, is that it introduces the notion of a two-dimensional world and the experience of a being that discovers the existence of a three-dimensional world which includes him and his partners [Rucker, 84].

The first approach to the fourth dimension (4D) was made by August Möbius in 1827. He speculated that rotations could work as reflections if any body (or figure) is passed through a higher dimension (one higher than the body or figure). For example, a right hand silhouette (a 2D figure) can be turned into a left hand silhouette passing it through the 3D space [Robbin, 92]. Möbius proposed that a 4D space is needed to turn right-handed three-dimensional crystals into left-handed crystals.

In England, Arthur Cayley and John J. Sylvester described an Euclidean geometry of four dimensions where hyperplanes are determined by noncoplanar quadruples of points. They were able to move into a higher dimension because they added a new axiom: "outside any given three-dimensional hyperplane, there are other points" [Banchoff, 96].

In 1854, George Bernhard Riemann broke the cult position that the Euclidean geometry had for two thousand years with the introduction of the theory of higher dimensions. In "*On the hypotheses which lie at the foundation of geometry*", Riemann exposed the novel properties of higher dimensional space and demonstrated that Euclid's geometry is based only in the perception [Kaku, 94].

In 1855, Ludwig Schläffi established that regular polytopes' boundary is composed by a finite number of solid cells (like polyhedra's boundary is composed by a finite number of polygons) in different hyperplanes and placed in the form that every cell's face is shared with another cell [Coxeter, 84]. Schläffi determined all six regular 4D polytopes and their numerical and metrical properties [Robbin, 92]. Unfortunately, the Schläffi's work doesn't have any illustration. In Chapter 2 are presented the properties of the hypercube (one of the 4D regular polytopes).

The first steps for the visualization of 4D polytopes were made in 1880's decade. In 1880, William Stringham presented for first time illustrations of many 4D polytopes in the *American Journal of Mathematics* [Robbin, 92]. In the same years, Charles Howard Hinton, in the Oxford University, published "*What is the fourth dimension?*" where presented three methods to visualize 4D polytopes: examining their shadows, their cross sections and their unravellings [Kaku, 94].

Finally in 1884, Edwin A. Abbot proposed a method to conceptualize the fourth dimensional space in his book *"Flatland"*, where a 2D being (A. Square) tries to understand 3D objects that appear to him by means of analogy (like we would try to understand 4D polytopes) [Rucker, 77]. Although the Abbot's method is considered one of the most effective, it was yet presented in the Plato's allegory of the cave.

1.2 Related Works

At the Bell Labs, in 1966, Michael Nöll created the first computer images of hypercubes. Important features of Nöll's programs were the use of stereo vision and 4D perspective projection ([Robbin, 92] and [Hollasch, 91]). The Nöll's method was the generation of the pictures via plotter and the transference onto film.

Thomas Banchoff (Mathematics Department at Brown University) has written computer programs that allow interactive manipulation (for example, with a joystick) of higher dimensional polytopes. Banchoff's technique of visualization is the projection of the polytopes' shadows onto 2D computer screens [Robbin, 92]. Banchoff and Charles Strauss are authors of the film *"The Hypercube: Projections and Slicing"*, which was presented at the International Congress of Mathematicians in Helsinki in 1978.

In [Hollasch, 91] it is mentioned the work of Scott Carey and Victor Steiner. They have rendered 4D polytopes to produce 3D "images" (like the rendering of a 3D object produces a 2D image). Finally, the results of 4D-3D rendering are 3D voxel fields.

[Banks, 92] presents techniques for interaction with 4D-surfaces projected in the computer screen. Banks describes the ways to recover lost information that the 4D-3D-2D projection causes by means of visualization cues like depth. Also, ten degrees of freedom in 4D space are identified (6 rotations and 4 translations) and the use of devices to control the interaction.

[Hollasch, 91] proposes a 4D ray-tracer that supports four-dimensional lighting, reflections, refractions, and also solves the hidden surfaces and shadowing problems in 4D space. The proposed ray-tracing method employs true four-space viewing parameters and geometry. Finally, the produced 3D field of RGB values is rendered with some of the existing methods.

1.3 4D Space-Time and Euclidean 4D Space

The concept of space-time (ST) was popularized with the apparition of the Albert Einstein's Relativity Theory and it is considered the most important innovation presented for him. For the relativists, time is considered as the fourth dimension [Russell, 84] and it is fully linked with space. Einstein proposed that time and space are not independent because an event must be described in terms of the place and the time at which it occurs [Kaku, 94]. For example, for a meeting it is necessary to specify a place in 3D space (a restaurant, a park, etc.) and the time (12:30 p.m., tomorrow, next Sunday, etc.). Consequently, space is an arbitrary 3D cross section of the 4D ST where 3D objects are moving forward in the direction of the remaining dimension, the time [Rucker, 77].

Time is not necessarily the fourth dimension. The Relativity Theory is an example were time is very useful as a higher dimension. Time must be considered as a fourth temporal dimension in the case of the 4D ST [Kaku, 94]. The fourth dimension is spatial, represented by a line perpendicular to each of three other perpendicular lines and it leads out of the space defined by the other three and never intersects them [Robbin, 92] (we have defined intuitively the Euclidean 4D space).

Newer physics theories are most interested in dimensions beyond the 4D ST defined by Einstein. For example, the Hyperspace Theory [Kaku, 94] proposes that ten dimensions (nine spatial and one temporal) are necessary to describe all the laws of nature in a simpler way and to unify all the forces of universe (gravity, electromagnetism and nuclear forces).

Finally, [Coxeter, 84] consider Euclidean 4D space as the space with four coordinates (x, y, z, w) instead of habitual two (x, y) or three (x, y, z). And it is established by him that two distinct points determine a straight line, three vertices of a triangle determine a plane and four vertices of a tetrahedron determine a hyperplane which has only a lineal equation that relates to the four coordinates.

From now on, when the term 4D space is referred, it must be interpreted as the Euclidean four-dimensional space.

1.4 Methods for Visualizing 4D Polytopes

It is known that one of the most important contribution of Charles Howard Hinton was the three methods to visualize 4D polytopes in our 3D space: examining their shadows, their unravellings and their cross sections. The method of the shadows consists in that if it is possible to make drawings of 3D solids when they are projected onto a plane, then it is possible to make drawings or 3D models of 4D polytopes when they are projected onto a hyperplane [Coxeter, 84]. Let us follow the analogy presented in "Flatland" [Abbot, 84]. If a 3D being wants to show a cube to a 2D being (a flatlander) then the first one must project the cube's shadow onto the plane where the flatlander lives. For this case, the projected shape could be, for example, a square inside another square (**Figure 1.1**).

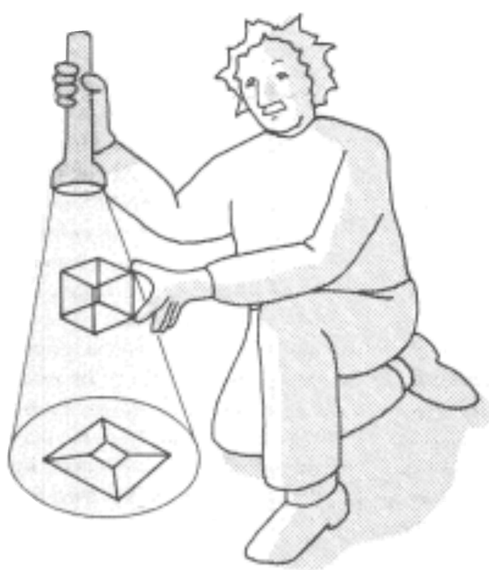


FIGURE 1.1
Projecting a cube on a plane [Kaku, 94].

If a 4D being wants to show us a hypercube, he must project the shadow onto the 3D space where we live. The projected body could be a cube inside another cube [Kaku, 94] called central projection [Banchoff, 96] (**Figure 1.2**). We know that a projected cube onto a plane is just an approximation of the real one. Analogously, the hypercube projected onto our 3D space is also a mimic of the real one.

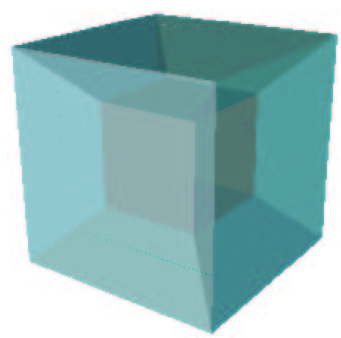


FIGURE 1.2
Hypercube's central projection onto the 3D space [Aguilera, 02b].

A cube can be unraveled as a 2D cross. The six faces on the cube's boundary will compose the 2D cross (**Figure 1.3**). The set of unraveled faces is called the unravellings of the cube. This is Hinton's second method for visualizing 4D polytopes.

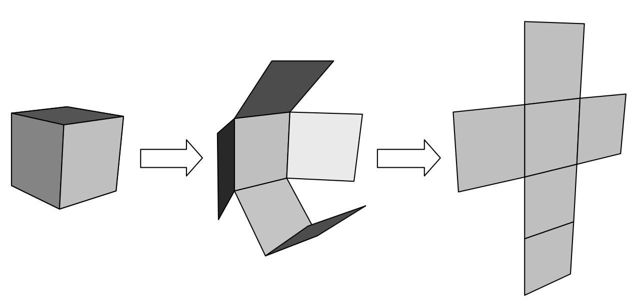


FIGURE 1.3
Unraveling the cube (own elaboration).

In analogous way, a hypercube also can be unraveled as a 3D cross. The 3D cross is composed by the eight cubes that forms the hypercube's boundary [Kaku, 94]. This 3D cross was named ***tesseract*** by C. H. Hinton (**Figure 1.4**).

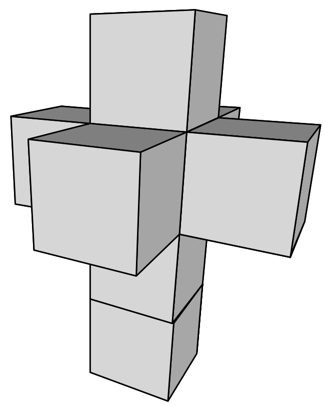


FIGURE 1.4
The tesseract [Aguilera, 02b].

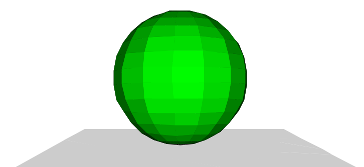
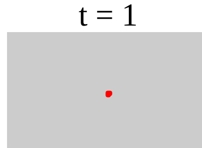
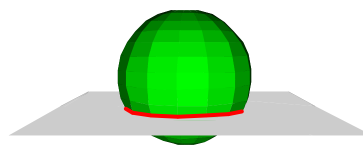
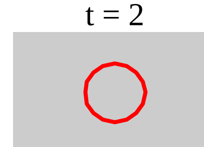
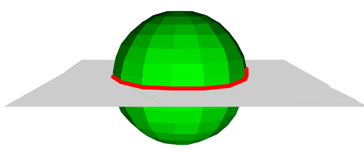
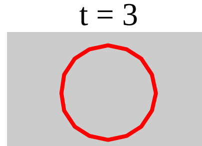
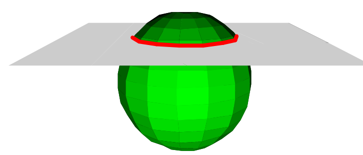
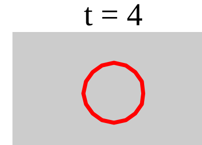
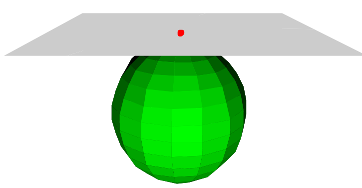
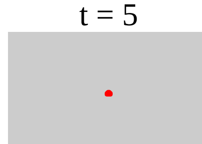
A flatlander will visualize the 2D cross, but he will not be able to assembly it back as a cube (even if the specific instructions are provided). This fact is true because of the needed face-rotations in the third dimension around an axis, which are physically impossible in the 2D space. However, it is possible for the flatlander to visualize the raveling process through the projection of the faces and their movements onto the 2D space where he lives.

Analogously, we can visualize the tesseract but we won't be able to assembly it back as a hypercube. We know this because of the needed cube-rotations in the fourth dimension around a plane which are physically impossible our 3D space. However, it is possible for us to visualize the raveling process through the projection of the volumes and their movements onto our 3D space. In chapter 5 will be presented the Aguilera and Pérez’s method for unraveling the hypercube and to visualize the process.

The slicing was the method used for Abbott in “Flatland” to describe the communication between 2D and 3D spaces [Banchoff, 96]. In that case, the 3D visitor, A. Sphere was perceived by A. Square (the flatlander) as a circle changing in size through the time (**Table 1.1**) because the first one was moving throught Flatland (a plane).

TABLE 1.1

Sphere' s plane intersections with Flatland (own elaboration).

3D view	Flatland view	3D view	Flatland view
	t = 1 		t = 2 
	t = 3 		t = 4 
	t = 5 		

In analogous way, if a 4D hypersphere visits our 3D space, we would see a point that increases its size in all directions to take the shape of a sphere (**Figure 1.5**). During these movements, the 4D hypersphere was moving though our 3D space and we visualize its 3D-slicings.

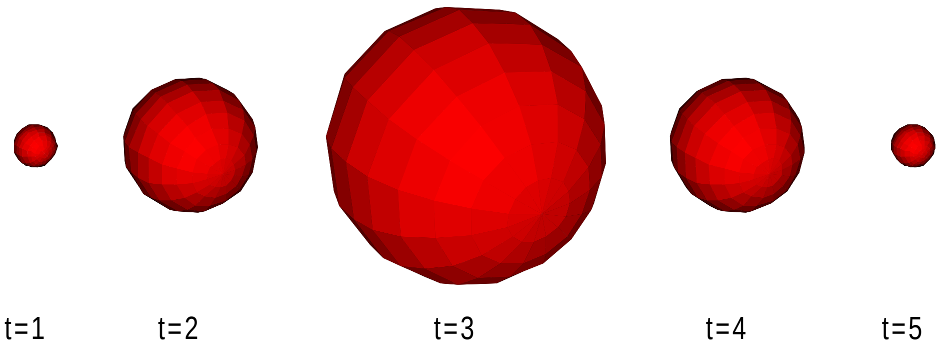
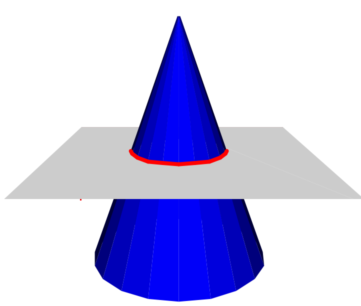
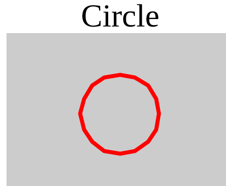
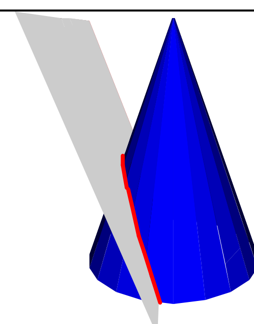
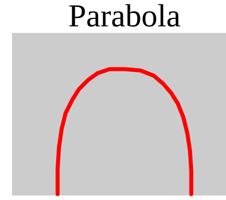
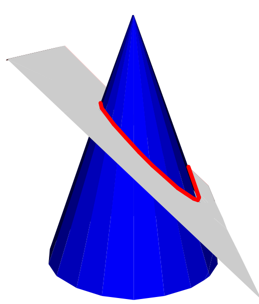
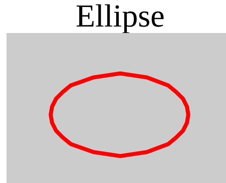
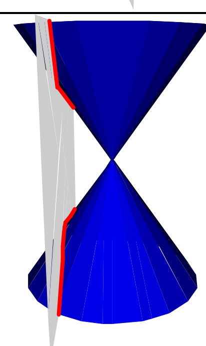
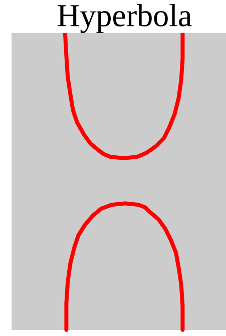


FIGURE 1.5
A 4D hypersphere seen in our 3D space (own elaboration).

The most common application of the slicings of a body are the conic sections. Slicing a cone we produce the hyperbola, the parabola, the ellipse and the circle (**Table 1.2**). A 2D slicing can be geometrically defined as the plane intersection with a body's surface. For analogy, a 3D slicing is the hyperplane intersection with a polytope's boundary.

TABLE 1.2
Slicing the cone and generating the conic sections (own elaboration).

3D view	Flatland view (Conic Section)	3D view	Flatland View (Conic Section)
	Circle 		Parabola 
	Ellipse 		Hyperbola 

1.5 Objectives

1.5.1 General Objectives

The general objective of our research is to propose and demonstrate how the numerical and geometrical properties for polygons (in 2D space) and for polyhedra (in 3D space) can be extended to define, in analogous way, the properties of 4D polytopes. Using these extensions, we will propose and demonstrate the generalizations that define the geometric and numerical properties of nD polytopes.

1.5.2 Specific Objectives

The study topics to be considered in our research are included, but not restricted to:

- 4D geometric transformations.
- Analysis and study of 4D-3D-2D projections.
- Numerical and geometrical properties of 4D polytopes.
- Boundary analysis for 4D polytopes.
- Modelling 4D polytopes.
- Boolean operations for 4D polytopes.

1.5.3 Coverage and Restrictions

- The study will be restricted to 4D Euclidean Space. As seen before, the 4D Euclidean Space is such whose points have four coordinates (x, y, z, w) juts like the 2D and 3D spaces have two coordinates (x, y) and three coordinates (x, y, z) respectively. Moreover, in this space, two distinct points determine a straight line, three vertices of a triangle determine a plane and four vertices of a tetrahedron determine a hyperplane which has only a lineal equations that relates to the four coordinates.

- Only it will be considered 4D Orthogonal Polytopes. A 4D Orthogonal Polytope is a polytope whose edges, faces and volumes (its boundary) are oriented in four orthogonal (perpendicular) directions to X, Y, Z and W axis of the 4D space.
- Specifically, this document will present our first experimental results about the 4D Orthogonal Pseudo-Polytopes' boundary. Based in those results we propose a set of generalizations that could be auxiliary in the study of nD Orthogonal Pseudo-Polytopes. We also present a method for unraveling the 4D hypercube, a procedure that is not specified, through transformations and their parameters, by some of the most known authors.

1.5.4 Organization

Besides this chapter, the structure for this document is the following:

- **Chapter 2: The Hypercube.** This chapter presents the numerical and geometrical properties of the most probably known 4D polytope, the hypercube (analogous to the cube in the 3D space).
- **Chapter 3: 4D Geometric Transformations.** There are presented the 4D translation, scaling and projections as extension of those applied in the 3D space. It is discussed one of the most interesting topic in the study of the 4D space, rotating around a plane.

- **Chapter 4: Unraveling the 4D Hypercube.** It is presented a method to hyper-flattening the hypercube's boundary, an analogous process to unravel a cube and getting a 2D cross.
- **Chapter 5: The 4D Orthogonal Polytopes.** This chapter is the kernel of this current work. It is an experimental and exhaustive analysis about the 4D Orthogonal Polytopes' boundary and their properties. Furthermore, there are proposed the generalizations of these properties to be applied in nD Orthogonal Polytopes.
- **Chapter 6: 4D Extreme Edges.** It is presented how 4D Orthogonal Polytopes can be analyzed by 3D methods, where the proposed results for these 3D analysis are the Extreme Edges.
- **Chapter 7: Results, Conclusions and Future Work.** Where is presented a resume of those proposed results in this work, their application in the prediction of some properties for 5D Orthogonal Polytopes and the lines of future research.