UNIVERSIDAD DE LAS AMÉRICAS PUEBLA

School of Engineering

Department of Industrial, Mechanical and Logistical Engineering

Analysis of Newton's Cradle, obtaining of the Damping Coefficient and the evaluation of its importance

Thesis presented to complete the requirements of the Honor Program by:

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Bachelor's in Mechanical Engineering

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Signature Sheet

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Dedicatoria

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Chapter 1

Introduction

In this project it will be studied the dynamic of a simple system with a complex behavior which is conformed by multiple and simultaneous collisions as well as by the effects of the hydrodynamic forces (damping coefficient).

This project was integrated by physical experiments treated with Image Processing and numerical simulations along with the performance and analysis of the Newton's Cradle.

Thereby, it would be possible to understand or re-understand in a better way the phenomenon of collisions, something that is more complicated and complex than one might think. To achieve the goal it was be necessary to utilize the existing theory along with *Image Processing* (this method will be explained further) and some simulations. With the results obtained, the damping coefficient could be computed. Later, throughout the document, it is going to be explained the impact this concept has in the argument that is trying to be demonstrated.

The Newton's Cradle is a device many people have learned something about conservation of energy and momentum with. And because the use of this device for this project was fundamental, one of the first and most curious impressions was that, because to obtain it was necessary to buy it through online stores, and while choosing amid the options, it was found that, at the comments section, previous buyers painted their purchase experience as bad. The most common assumption was that the cradle was broken just because in a very "short" range of time, the balls started to stop. Thus, they believed the product was a swindle because they expected the balls to swing endlessly or for at least, a larger period of time.

This misconception of the event arise from was it is taught generally in schools; for example. But, actually, the collision of bodies and in this specific case, the Newton's Cradle, represents more than a simple and "ideal" contact between bodies.

In order to present a lot of features that takes part and construct the concept of *collision*, physical experiments and subsequently some numerical experiments that involved simulations were performed; their results were evaluated, analyzed and compared with the simulations to drive some conclusions about this whole situation and to determine if the results developed in the real life differ from a whole ideal ambiance or if the results are similar.

Firstly, as the theoretical framework, the description of what a pendulum is and how does it work, how many types of pendulum, the historical background is presented. Then, the description of the Newton's Cradle and its functioning are displayed as well.

After that, some important aspects that are involved and that are very important about the contact of bodies, were described taking into consideration the environmental characteristics in which the pendulum performs, the material and the geometry of the spheres and the dissection of what happens once two spheres come into contact.

In *Chapter 3*, the process of the physical experiments and of the simulations are explained from its methodology and construction till the results and their justification; thereafter, before all else, in case of the physical experiments, there were declared the most important elements and characteristics of the experiment, and its how the experiments were performed withal including the treatment the results received in order to understand the information they provided.

Finally, in *Chapter 4*, the analysis of both the simulations and physical experiment results was realized with its respective explanation or justification which culminates into the conclusions and final thoughts in *Chapter 5*.

1.1 Justification

In general terms, there is a misconception around what happens when a body enters in contact with another; this confusion would generally come from the school; when, at trying to make the students to understand (probably) in a faster or non-complicated way, they learn, either incomplete or wrong concepts of this topic. The Newton's Cradle is a very good example of this statement; but also is a good way to try mending this matter. In this project it will help to understand, firstly, how a single pendulum works and also how the multiple contact works.

This task is important because the pendulum and the multiple contact have wide applications and a large importance in our daily life; for example, in the construction sector, the pendulum has a fundamental role, whereas the multiple contact of bodies is present in most every machine no matter how elementary or complex it is; if it experiences vibrations, it is of the concern of the multiple contact of bodies. Throughout the document, this applications will be better explained.

And then, this work is important because hereby it will be tried the capacity of knowing how to present a case, explore the options to solve it and finally, join the theoretical information with the experimentation either it is physical or with simulations.

1.2 Objectives

- Demonstrate the complexity of the concept of the multiple bodies contact
- To be capable to properly apply the knowledge learned throughout my studies in this University.
- To analyze correctly the experiments performed via physical and numerical in order to determine if what happens in real life agrees with the theory.
  - Demonstrate if considering the ambiance hydrodynamic resistance (air); where the Newton's Cradle performances, should be taken into account in order to analyze the device behavior.

Chapter 2

Theoretical Framework

2.1 Simple Pendulum

According to the Merriam-Webster dictionary, a simple pendulum consists of a bob suspended at the end of a thread sometimes it can be considered massless in order to make the calculations easier (Merriam-Webster, 2020).

In a pendulum, the body is attached to a fixed point, so it can swing under the effect of some forces, just like gravity, the tension of the thread and other that will be explained later. The pendulum pursues an angular motion because the body will be forced to move back and forth or side by side (depending the point of view). At some stages of the path, the massless body has different heights, and so, its velocity will change whereas swinging. The long the body takes to swing from where it is released until it returns to the same point, it is known as period. Something interesting is that, at low altitudes and at the Earth's poles, the period will be shorter compared with high altitudes and at the Equator (Man, 2014).

The duration of time the pendulum takes to complete a cycle swings is determined by the length of the element the pendulum is attached to and for small oscillations the period is independent from the amplitude; even there were a change in the mass of the bob. Along the sections, this whole concept will be explained more in detail.

The pendulum has wide applications, for example it has been used to adjust the movement of the clocks or to exactly measure the acceleration (Kater's pendulum). It has also been applied as a metronome mechanism (used to keep the same speed of the music). Even for the human recreation time, the pendulum is present; for instance, in most of the parks, at least, there is a swing. The pendulum's theory is also requested in structure statics for zones with high seismic activity. In some buildings of Chile and in The San Francisco International Airport Terminal, a mechanical device known as Frictional Pendulums is employed to allow the structure to "sway" with the movement of the ground so the building is prevented from falling or to avoid some damages in case of an earthquake (Abel, 2018).

2.2 Types of pendulums

- Simple pendulum

  The description of this type of pendulum is the section above.

- Foucault Pendulum

  This pendulum named after the French physicist León Foucault was developed around 1850 to demonstrate the rotation of the Earth. Its action is a result of the Coriolis effect. This was the first laboratory demonstration of the Earth spinning (Encyclopaedia Britannica, 2017).

- Double Pendulum

  This pendulum operates with small oscillations demonstrating the phenomenon of beats. As the energy increases, the oscillations become chaotic. Even the phenomena can be described by a number of ordinary differential equations, the chaos matter, the Lorenz system has a similar model where three equations so the chaotic behavior can be described (Neumann, 2016).

- Kater's pendulum

  It is a physical pendulum with two adjustable knife edges for an accurate determination of the Earth's gravitational force. When adding a second knife-edge pivot and two adjustable masses to the physical pendulum the value of the gravity has a 0.2% precision (Harvard University, no date).

- Schuler pendulum

  It is a fictitious pendulum where the radius of the Earth becomes the string and the mass is the center of the Earth. Its oscillation period is of 84.4 minutes. This is a design used in inertial navigation systems such as submarines, ships or aircraft because the curvature the surface of the Earth must be considered (Huber., Borges, 1983).

- Spherical pendulum

  A spherical pendulum is a type of simple pendulum conformed by a punctual mass suspended by a wire to a fixed point with a inextensible string with a certain length. The mass is capable to move in all direction while the strain persists stiffed (Fitzpatrick, 2011).

- Ballistic pendulum

  It is a device used to measure the velocity of a projectile (like a bullet) and in consequence, it can be used also to measure momentum by determining it from the amplitude of the pendulum. It was invented around 1750 by a French military engineer (Nave, no date).

- Conical pendulum

  Is a famous device quite similar to a simple pendulum but instead of swinging back and forth, it moves at a constant speed in a circular path with the string tracing out a cone. It was generally used to regulate the speed of steam engines around 1800's. The tether-ball also uses the same principle (More, 2020).

2.3 History of the pendulum

- Zhang Heng

  One of the first registers of pendulum applications dates around 132AD in China. *Zhang Heng* was a Chinese scientist, astronomer, painter and writer who invented the *Houfeng Didong Yi.* An apparatus designed to predict where an earthquake could come from. The mechanism consisted on a big bronze container resembling an urn decorated with four dragon heads each one placed with a separation of ninety degrees, indicating the north, south, west and east directions. Inside the urn, there was a pendulum connected with every head. When the pendulum started oscillating (the earth started to move) a bronze ball would came outward one of the four dragon heads into the mouth of another metal object shaped as a toad (Hsiao, 2009).

- Galileo Galilei

  According to some information, around the end of seventeenth century, when Galileo was approximately twenty years old, he was attending a lamp swinging from a cathedral ceiling. During the examination, he needed to measure the time with his own pulse and realized that no matter how big or small the amplitudes were, their period was practically the same since the lamp returned to a standstill with an equal duration of time. He was the one who discovered the law of the pendulum which would be used later in the construction of clocks, as it could be used to regulate them (Bellis, 2017).

- Christiaan Huygens

  Christiaan Huygens was a dutch scientist, astronomer, physicist, mathematician and inventor. Famous for discovering the shape of Saturn's rings, the moon Titan, the wave theory of light, the formula for the centripetal force and for discover the way the pendulum clock worked (Helmenstine, 2019).

In 1665, Huygens, detected that two pendulums (from two identical pendulum-clocks), were always going to oscillate in synchronicity. Nearly five years ago, scientists from Mexico and Eindhoven validated this premise about synchronicity which was developed for practically over four centuries ago. The experiment was published by *Scientific Reports* in 2016 (Peña, Olvera, Nijmeijer & Álvarez, 2016).

This demonstration consisted on the exact replication of the phenomenon Huygens witnessed many years ago. Two pendulum clocks were specially constructed as monumental pendulum clocks with wooden structure. It was reported that the pendulums presented a phase synchronized motion which at first, can be contradictory with Huygen's observation because it is said that what he specifically saw was an anti-phase synchronized motion even though, he could have possibly known he witnessed an in-phase synchronized motion. Besides, according to their experiments, when clocks are synchronized, their oscillation frequency decreases a bit.

2.4 Simple Pendulum System Description

A simple pendulum which for this case to analyze, is constituted by a point mass hung from a string and ruled by simple harmonic motion. The mass is attached to the string but, the mass can oscillate (or swing) in the space if an **impulse** is applied. If uninhibited by friction or by any other dissipation of energy, its motion would continue indefinitely.

This case is an specific representation of a simple harmonic oscillator, which consists on a system that once disturbed from its equilibrium, it experiences a restoring force proportional to the displacement. This phenomenon obeys Hooke's Law.

$$F = -kx \tag{2.1}$$

Where k refers to the constant of restitution and x to the displacement.

In Newton's Mechanics, the Newton's Second Law stipulates that the force is equal to the mass times the acceleration.

$$F = m\frac{d^2x}{dt^2} \tag{2.2}$$

And equalizing Equations 2.1 and 2.2, and employing the concept from the Simple Harmonic Motion definition where $a(x) = -\omega^2 x$; the equation of motion, which is a second-order linear ordinary differential equation, can be obtained:

$$\begin{gathered} F = m\frac{d^2x}{dt^2} = -kx \\ \text{So} \\ \frac{d^2x}{dt^2} = \frac{-kx}{m} \end{gathered} \tag{2.3}$$

$$\begin{aligned} F &= -m\omega^2 x \\ -kx = -m\omega^2 x\omega &= \sqrt{k/m} \end{aligned} \tag{2.4}$$

$$\omega = \sqrt{k/m} \tag{2.5}$$

Then, the general solution can be determined:

$$x(t) = c_1 cos(\omega t) + c_2 sin(\omega t) \tag{2.6}$$

The constants (c_1 and c_2) are found if the time is established as $t = 0$ so $x(0) = c_1$ meaning that c_1 expresses the initial position. c_2 refers to the initial speed since: $\dot{x}(0) = \omega c_2$ (Russel, 2011). Finally, Equation 2.6 can be also written as:

$$x(t) = A\cos(\omega t - \varphi) \tag{2.7}$$

Where A refers to the amplitude, ω to the angular frequency, t to time and φ to the initial phase.

The expressions for period and frequency can be represented as:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{\frac{m}{k}} \tag{2.8}$$

$$f = \frac{1}{T} = \frac{\sqrt{\frac{k}{m}}}{2\pi} \tag{2.9}$$

2.5 Small-angle Approximation

A simple pendulum can be defined as a simple harmonic oscillator if the restoring force is directly proportional to the displacement. To analyze this device, the Newton's Second Law can be useful in order to formulate an equation for the forces acting on the pendulum. The next analysis and its equations can only be satisfied for a small swing angle i.e. it is only valid for an angle $\theta \leq 20°$ (Russel, 2018).

Taking as reference Figure 2.1, the x component of the weight will control the oscillating motion and so forth, the restoring force on a pendulum is:

$$F_r = -mgsin(\theta) \tag{2.10}$$

And the y component will be the centripetal acceleration (tension):

$$\begin{aligned} F_c - mgcos(\theta) &= 0 \\ F_c &= mgcos(\theta) \end{aligned} \tag{2.11}$$

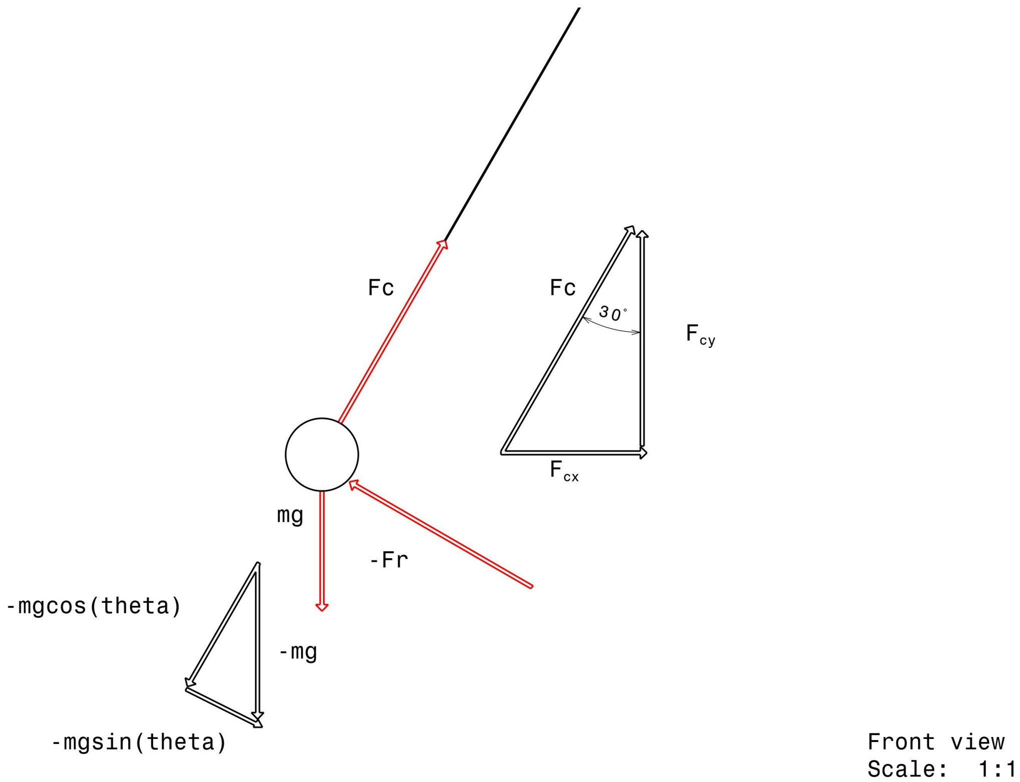


Figure 2.1: Force diagram for a simple pendulum.

So, since these components are accelerations ($F = ma$) they can be also expressed as:

$$a_x = \frac{F_c sin(\theta)}{m}$$
$$a_y = \frac{F_c cos(\theta)}{m} - g \tag{2.12}$$

Equation of motion for rotative systems:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0 \tag{2.13}$$

To simplify the solution, the small-angle approximation can be used. It is known that for small angles, $sin(\theta) = \theta$ and that $cos(\theta) = 1 - (\frac{\theta^2}{2})$.

Then, the components of the force F_c are:

$$F_r = -mg(\theta) \quad \text{and} \quad F_c = -mg(1 - (\frac{\theta^2}{2})) \tag{2.14}$$

2.5.1 Simple Pendulum with Friction

Considering the displacement of the pendulum as curvilinear motion and friction an a external force, the forces diagram would be:

$$s = l\theta \tag{2.15}$$

$$a = l\ddot{\theta} \tag{2.16}$$

Where s represents the angular displacement since the path the sphere follows, is an arc.

If this system is considered as a Damped Harmonic Oscillator, the compiling of the conditions is the next:

Frictional force:[I]

$$F_f = -c\frac{ds}{dt} = -c\ l\ \dot{\theta} \tag{2.17}$$

Where c represents the viscous damping coefficient, $\frac{ds}{dt}$ the variation of the angular displacement, l is the length of the string and $\dot{\theta}$ the angular velocity.

And in the case of a external force:

$$F_e = F_o \cos \omega t \tag{2.18}$$

[I]This equation is also commonly identified as: $F_f = -c\ v$

The balance of these equations of the system will be :

$$ml\ddot{\theta} + c\ l\dot{\theta} + mg\sin\theta = 0 \tag{2.19}$$

It can be possible to non dimensionalize, considering that $\omega = \sqrt{\frac{g}{l}}$:

$$[ml\ddot{\theta} + c\ l\dot{\theta} + mg\theta = 0] * [\frac{1}{ml}] \tag{2.20}$$

It can also be expressed a dimensionless time by $t_d = \omega_0 t$

And the dimensionless equation will be:

$$\frac{d^2\theta}{d\tau^2} + \alpha\frac{d\theta}{d\tau} + sin\theta = \gamma\cos\beta\tau \tag{2.21}$$

Where $\alpha = \frac{c}{m\omega_0}$, $\beta = \frac{\omega}{\omega_0}$ and $\beta = \frac{F_0}{ml\omega_0^2}$ which are dimensionless parameters due the presence of a "sin" term which makes the equation to be a non linear equation. The α term represents the damping, the γ term the forcing amplitude and β term the forcing frequency.

Considering the theory of the small-angle approximation, Equation 2.21 can be rewritten and define a new *theta* with $\Theta = \frac{\theta}{\gamma}$ (Chasnov, 2019).So, at the end, it is left:

$$\frac{d^2\Theta}{d\tau^2} + \alpha\frac{d\Theta}{d\tau} + \theta = \cos\beta\tau \tag{2.22}$$

2.6 Exact Solution

To solve the final equations in the above section, numerical methods must be employed in order to compute and get the results since the analytical solution is arduous. The nature of the equation is of an elliptical integral where the variables are: time (t) and the angle (θ).

As seen above, the equation of motion of the sphere takes place along the x axis and applying the Second Newton's Law for rotational systems can be described as:

$$\tau = I\alpha = mL^2\frac{d^2\theta}{dt^2} \rightarrow = -mgsin(\theta)L \tag{2.23}$$

So

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}sin(\theta) = 0 \tag{2.24}$$

The simple harmonic solution is:

$$\theta(t) = \theta_0 cos(\omega t) \tag{2.25}$$

Where θ_0 is the angular displacement and $\omega = \sqrt{\frac{g}{L}}$ [II]

The period it is independent from the mass of the ball but it does depend on the length of the string. The larger the string is, the larger the period is.

What happens in reality, is that, when the angle is not small enough, the equation of motion it is not longer linear.

[II]In Subsection 3.2.1 it will be explained this concept.

2.6.1 Numerical Solution

This function helps to solve an ordinary differential equation (which means that any equation of this kind must have constant coefficients with initial conditions and must be non homogeneous.) This function in particular, only solves a first grade "ODE" (Ordinary Differential Equation).

The manual solution consists on a change of variable and the same happens with this method but a vector is used instead of a scalar. It can be also useful to perform simulations of solutions that change in time in a single degree of freedom system (such as the pendulum).

Due the mechanisms a pendulum follows, a differential equation can be determine and by applying the Second Newton's Law, where mass times acceleration is equal to $-mgsin(\theta)$.

$$\begin{aligned} a = l\frac{d^2\theta}{dt^2} \\ ml\frac{\partial^2}{\partial t^2\theta}t = -mgsin(\theta(t)) \end{aligned} \tag{2.26}$$

And considering over again the definition of the natural frequency being the square root of the ratio between the gravity and the length of the pendulum, the natural frequency is defined:

$$\frac{\partial^2}{\partial t^2} = -(\omega_0)^2 sin(\theta(t)) \tag{2.27}$$

Because this is a non linear equation, a Taylor method with the expansion of $sin(\theta)$ can be useful. And since:

$$\frac{\partial^2}{\partial t^2}\theta(t) = -\omega_0^2\theta(t) \tag{2.28}$$

is also needed to display how the path is going to be ranged. Due the mechanism is a pendulum, it will oscillate. In this case, the concept of $\omega_0 t$ is called *phase*, owing to the *cosine* and *sine* functions repeat each cycle of 2π.

The period needed for the phase to change every cycle, will be the period.

The other aspects needed to display are the conditions, such as gravity and the length of the pendulum.

Is important to emphasize that, energy must be conserved (kinetic and potential energy)

$$E = \frac{1}{2}mr^2(\frac{d\theta}{dt})^2 + mgr(1 - \cos(\theta))1 - \cos\theta = 2\sin^2(\frac{\theta}{2})$$

and

$$\omega_0 = \sqrt{\frac{g}{l}} \tag{2.29}$$

$$\frac{E}{mr^2} = \frac{1}{2}[(\frac{d\theta}{dt}^2) + (2\omega_0 \sin\frac{\theta}{2})^2]$$

Through the software *Matlab*, the system can be solved with the Function *ode45* which is a function useful to solve non-stiff differential equations with a medium order method. In the next section, a part of the solution is shown.

Part of the Solution employing *ODE 45* with Matlab

T term indicates kinetic energy and V term indicates potential energy.

$$\begin{aligned}
\triangle T + \triangle V &= 0 \\
\triangle T &= -\frac{1}{2}I(\frac{d\theta}{dt})^2 + \frac{1}{2}I(\frac{d\theta}{dt})^2 \\
\triangle V = mg \triangle h &= -mgl[cos\theta - cos\theta_0] \\
\frac{1}{2}ml^2(\frac{d\theta}{dt})^2 - mgl[cos\theta - cos\theta_0] &= 0 \\
\frac{d\theta}{dt}^2 &= 2\frac{g}{l}[cos\theta - cos\theta_0] \\
\frac{d\theta}{dt} &= \pm 1\sqrt{\frac{2g}{l}cos\theta - cos\theta_0} \\
\frac{d\theta}{dt} &= -\sqrt{\frac{2g}{l}}(cos\theta - cos\theta_0)^{\frac{1}{2}} \\
\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{cos\theta - cos\theta_0}} &= -\sqrt{\frac{2g}{l}}\int_0^t dt \\
\frac{T}{4} &= \sqrt{\frac{l}{2g}}\int_0^{\theta_0} \frac{d\theta}{\sqrt{cos\theta - cos\theta_0}}
\end{aligned} \tag{2.30}$$

First, it is required to derive the equation of motion and then:

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}\theta(t) &= \frac{-g\sin(\theta(t))}{l} \\
\omega_0 &= \sqrt{\frac{g}{l}} \\
\frac{\partial^2}{\partial t^2}\theta(t) &= -\omega_0^2\sin((t))
\end{aligned} \tag{2.31}$$

The total energy must be conserved (potential and kinetic)

$$\begin{aligned}
E &= \frac{1}{2}l^2(\frac{d\theta}{dt})^2 + mgr(1 - \cos\theta) \\
\text{Because } 1 - cos\theta &= 2sin^2(\frac{\theta}{2}) \\
\frac{E}{ml^2} &= \frac{1}{2}[(\frac{d\theta}{dt})^2 + 2\omega_2\sin\frac{\theta}{2}]
\end{aligned} \tag{2.32}$$

2.7 Newton's Cradle

2.7.1 History

Edme Mariotte

The first studies explaining how this device works were expounded around 1670 under the authorship of Edme Mariotte. He was quoted in Newton's Principia along with other collaborators who are considered as pioneers in experimenting on collisions of pendulum balls. *Besides he was the first one who separated the collisions into two kinds (elastic and inelastic collisions) and also the first one who showed that during impact, elastic bodies such as steel or glass, actually deformed* (cite taken from the Museo Galileo - Istituto e Museo di Storia della Scienza, 2020).

The contribution of Wallis, Wren, Huygens and Newton defined how dynamics work into the well known concept of "momentum" which it is simply the quantity of motion (momentum) or mass times speed.

In his treatise published in 1673 and titled: "*Traité de la percussion ou choc des corps*" (Treatise on percussion or shock of bodies) there was included the explanation of the phenomenon arguing that when having a pendulum conformed by four balls (see Figure 2.2), if a determined inclination is given to the wire of a ball located at the end of the device, and then the ball, is released. When it contacts its nearest ball, it will eventually stop while the energy will be transmitted to the remaining balls, which are going always remain immobile; but the ball located at the other end of the cradle will move with the same energy and consequently, with the same speed of the first ball. Which means that the energy is going to be conserved always during the impact (Cross, 2012).

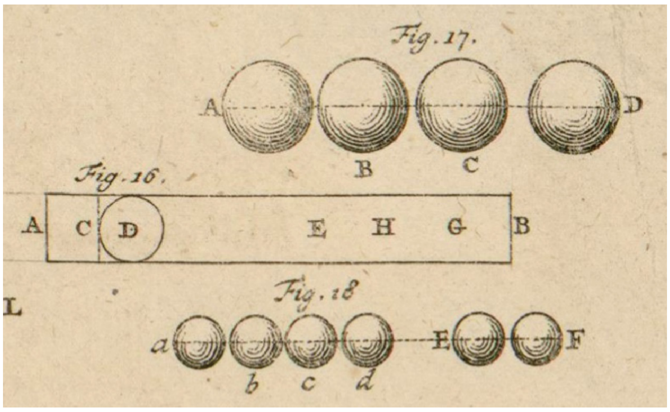


Figure 2.2: Four-ball pendulum designed by Edme Mariotte (Cross, 2012).

He also claimed that the number of balls emerged is always equal to the number of incident balls.

Mariotte was a French physicist who specialized in the mechanics and elastic properties of gases. Also conducted experiments on air resistance, barometry, hydrodynamics and discovered the "blind spot" of the eye. He additionally formulated the law on the compressibility of gases discovered independently by Robert Boyle.

Christiaan Huygens

Christiaan Huygens was a Dutch scientist, astronomer, physicist, mathematician, horologist and inventor. Famous for discovering the shape of Saturn's rings, the moon Titan the light wave theory, the formula for the centripetal force and for inventing the pendulum clock. In his theory of impact (Mahoney, 1995), he states:

- Two equal bodies collide directly when the motion and the contact occur on the straight line joining their centers of gravity
- When two bodies collide with one another, even if both together are further subject to another uniform motion, they will move each other with respect to a body that is carried by the same common motion no differently than if this motion extraneous to all were absent. So, if someone conveyed on a boat that is moving with an uniform motion were to cause equal balls to strike one another

at equal speeds with respect to himself and the parts of the boat, both should rebound also at equal speeds respect to the passenger,alike it would happen if he were to cause the balls to collide at equal speeds in a boat at rest or while standing on the ground

- If a larger body crashes with a smaller one at rest, it will give some of its motion and thus it will lose something of its own
- Proposition I: If a body at rest is hit by another identical body, after contact the latter will be at rest and the body at rest will achieve the speed correspondent to the hitting body
- Proposition II: If two identical bodies moved at different speeds strike each other, they are going to be moved with exchanged speeds after their contact
- Proposition III: A body with any large will change its position by colliding a moving body no matter how small it can be at any speed.

2.7.2 Newton's Cradle System Description

Usually, for the demonstration of the momentum conservation, the Newton's Cradle is employed. The Newton's Cradle is a device composed by a frame with wires (commonly made of nylon). The wires have of the same length.

From the wires, five metal spheres (normally made of steel) are hung from. The balls have the same geometrical and physical dimensions (mass, density). The distance between the center of the spheres is the diameter of the sphere. See Figures 2.3and 2.4

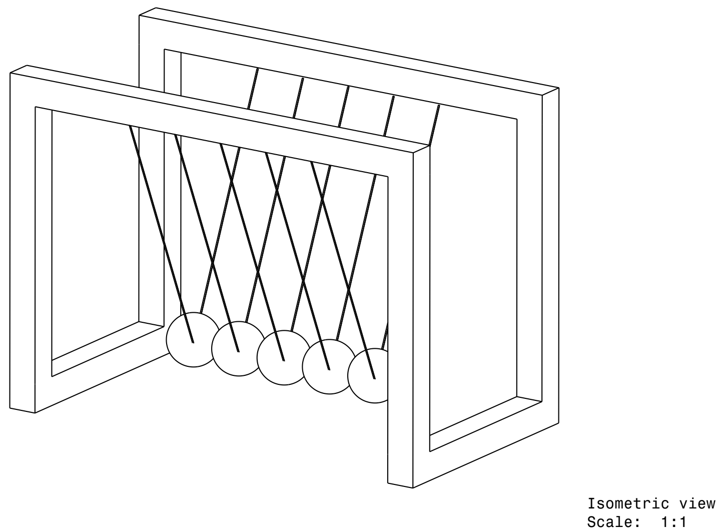


Figure 2.3: Isometric view of the Newton's Cradle used in the experiments.

The way this apparatus work, consists in two simple steps: raising one of the two balls place at the extremes and then, releasing it. The ideal process would be the momentum (and thus energy) to be conserved and the translation of the spheres were delimited to be necessarily a *2D* translation.

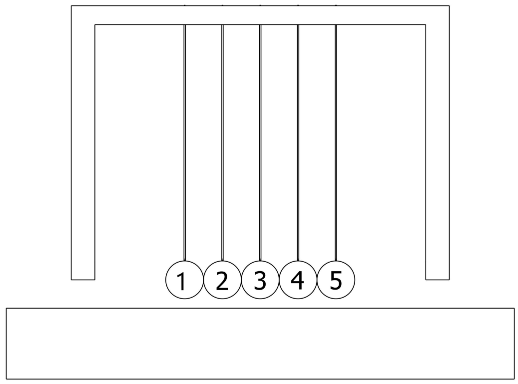


Figure 2.4: Diagram of the most popular version of the Newton's Cradle.

The typical illustration taught is that, the discharged ball as soon as it makes contact (collide) with the others, it abruptly stops due the energy transmission and so, the sphere collided which receives the energy, it transmits it to the remaining spheres. Until the energy, reaches ball *5*. The *5* sphere should starts moving with the same velocity and the same direction the sphere *1* had at first. So in summary, at the beginning, the sphere *1* is released and the energy spreads along balls *2*, *3* and *4* to reach ball *5* so it this can move but the previous balls, stay still. See Figures 2.5 and 2.6

There is a model which tries to explain the mechanism. It pretends to see the whole set of steady spheres as a single body. But if each ball is considered to have a mass (m) of mass, the body will be $4m$. Therefore, at the collision, a mass (m) (released sphere) will hit the ($4m$) mass (steady balls). And also considering that the collision is elastic, the final velocity of ball *1* should be $\frac{-3v_0}{5}$ and others velocity should be $\frac{2v_0}{5}$. Thus, the first sphere should bounce with three fifths of the original velocity and the others would start to swing with two fifths of ball *1* velocity had originality. Even with this model, momentum and energy can be conserved perfectly, it cannot be accurate since it is not quite realistic. (Gavenda, 1997)

This model is not accurate since the balls are not fastened with each other, each one has its own independent final velocities v_2, v_3, v_4 and v_5. That is why it cannot be assumed all of them would have the same final velocity.

It also has to be considered that the spheres have finite size and elastic properties and thus, momentum and energy propagates in a finite interval of time.

Is unknown who had the idea of hanging balls from crossbars or when it was first tried but the responsible for its name "*Newton's Cradle*" was the English actor, Simon Prebble who named it in honor of Isaac Newton. But actually, Newton was not the first scientist on examining this phenomenon or not even invented the cradle, in fact, more than twenty years before he had published "*Pilosophiae Naturalis Principia Mathematica*", the Dutch physicist Christiaan Huygens, described the theoretical principles of how a pendulum works. Specifically, he was the one who noted the conservation of momentum and of kinetic energy except that he did not apply this last concept but indicated it as if it were a quantity data.

The phenomenon, in simple terms consists in raising a ball with a mass m assuming the raising ball is the number one. Illustrated in figures Figure 2.5 and in Figure 2.6.

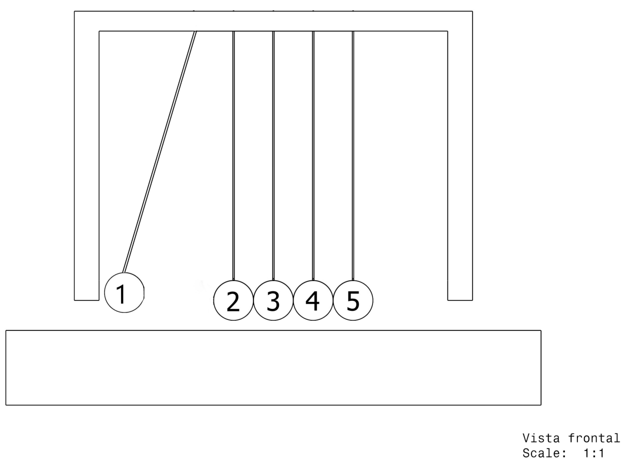


Figure 2.5: Illustration of the Newton's Cradle performance.

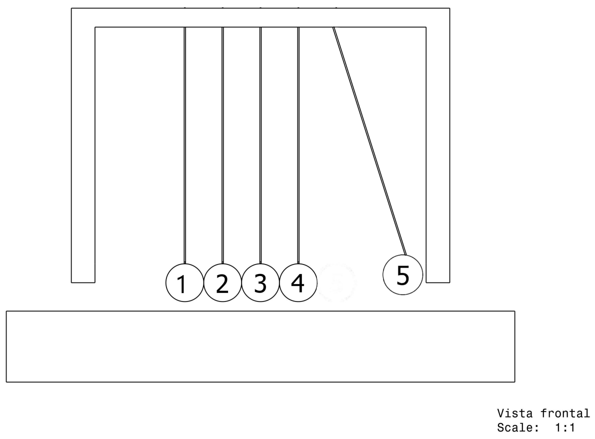


Figure 2.6: Illustration of the Newton's Cradle performance.

At that moment, the velocity of all the balls is zero v_0. The only difference between the ball *1* and the others, at the beginning, is that the ball *1* has potential energy (assuming the reference plane is fixed at the center of mass of the balls.

After raising it, the ball is dropped.

Due to the conservation of energy, in this case, momentum, balls *2*, *3* and *4* will remain at rest, hanging still. But the *5* ball will be the only one which is going to jump forward. When the ball (*5*) comes back and collide, all the five balls are momentarily reunited before the ball *1* is pushed away over again.

2.8 Physics Concepts

2.8.1 Drag Force and Reynolds Number

Whenever, no matter what, in a flow stream, there is an obstacle or an object moving through the flow, a resistance will be produced. That resistance is known as *Drag Force*. Therefore, the molecules of the fluid near the object are naturally disturbed. That is when aerodynamics forces are generated and their magnitudes depend on the shape, speed of the object, the mass, viscosity, compressibility or springiness of the fluid. (National Aeronautics and Space Administration, 2015)

Aerodynamic forces are generated where dynamics are present. If an object is moving through the space (a fluid), the molecules nearby the body are disturbed and start moving around it.

The relation between the aerodynamic forces and the viscosity can be understood when the object moves (assuming air is the fluid), the molecules will get stuck into the surface of the object, so there will be a kind of coat or layer of air around it, that eventually, will change the shape of the object, and so, the flow of the fluid will react with the edge of this "new boundary". However, the layer will get separated from the shape and can build a new shape which can be completely different from the physical shape; and at the same time, the flow conditions are unstable because the flow change with respect to time.

That is why the "new boundary" (also known as Boundary Layer) is critical to determine the drag of an object.

Since the big deal lays on the viscosity, there is a parameter called "Reynolds number". Which is the relation between inertial and viscous forces. In other words, it will be the relation between the motion and through a fluid which can be or not be heavy and gluey.(National Aeronautics and Space Administration, 2014)

It is also important to remind that there are two types of flows: Laminar and turbulent.

A laminar flow has a quite ordered motion defined by smooth streamlines and generally at low velocities. Like oils because they are highly viscous.

A turbulent flow has a quite disordered motion and a region where the velocity will always vary. And frequently, this flow exists when high velocities are present and the fluid has a low viscosity.

A way to determine if a flow is going to be turbulent or laminar, is employing the Reynolds number. A low Re will mean a laminar flow while a high Re will mean a turbulent flow.

If Re $<$1: the flow is laminar

If Re $>$1: the flow is turbulent

For a big Re, the inertial forces, which are proportional to the fluid density and the square of the fluid velocity, are larger in comparison to viscous forces, and thus the viscous forces cannot prevent the random and rapid fluctuations of the fluid.

For a small Reynolds number, the viscous forces can suppress these fluctuations and to keep the fluid "straight".

$$Re = \frac{Inertial force}{Viscous force} = \frac{ma}{\tau A} = \frac{\rho A d \frac{du}{dt}}{\mu \frac{du}{dy} A} = \frac{\rho d \frac{dy}{dt}}{\mu} = \frac{\rho d U_o}{\mu} = \frac{d U_o}{\nu} \tag{2.33}$$

Where:

- U_o is the characteristic flow velocity $[m/s]$
- d is the characteristic linear length $[m]$
- ρ is the density $[kg/m^3]$
- μ is the dynamic viscosity $[kg/ms] = Ns/m^2 = Pa\ s$
- ν=kinematic viscosity $[m^2/s]$

$$Re = \frac{\rho U_o L}{\mu} \tag{2.34}$$

Where:

- L is the traveled length or the diameter of the system $[m]$
- $\rho U_o L = \frac{kg}{m\ s}$
- $\mu = \frac{kg}{ms}$

The Reynolds number is an dimensionless number because the units on the numerator and the denominator are the same. Multiplying each other by vt, at the end the Reynolds number will have force dimensions.

$$\frac{\rho\ U_o\ L * U_o L}{\mu * U_o L} = \frac{\frac{kgm}{s^2}}{\frac{kgm}{s^2}} = \frac{N}{N} \tag{2.35}$$

2.8.2 Conservation of Momentum through Integral Equation Approach

Newton's Second Law states that the rate of change of time for a linear momentum, is proportional to the sum of all the applied forces.

$$\frac{D}{Dt} = (m\vec{u}) = \frac{D}{Dt}[\int_\nu \rho\vec{u}d\nu = \sum vecF \tag{2.36}$$

Reynolds transport theorem

$$\frac{D}{Dt}\int_\nu = \rho\vec{u}d\nu = \frac{\partial}{\partial t}\int_\nu(\rho\vec{u}d\nu + \int_S(p)\vec{u} \tag{2.37}$$

Integration equation of conservation momentum

$$\frac{\partial}{\partial t}\int_\nu(\rho\vec{u})d\nu + \int_S(\rho\vec{u})\vec{u_r}\cdot\vec{n}dS = \sum\vec{F} \tag{2.38}$$

2.8.3 Drag Force

Drag, according to the Britannica Encyclopedia, is the force a flowing fluid applies into a body or into a hindrance standing on its path. Whether this force is big or small partly depends on viscosity .(Encyclopædia Britannica, 2009)

A fluid stream applies this force to any kind of obstruction or applied by an object moving trough a fluid. The calculation of that force is determining for the design of vehicles, ships, suspension bridges, cooling towers, etc.

This force is not provoked by a force field (gravitational or electromagnetic). Necessarily, a fluid and motion are required so there drag can exist. Since it has direction (acts opposite to the body's motion) and magnitude, it is a vector and it can be also understood as the resistance of motion through a fluid. (National Aeronautics and Space Administration, 2015)

$$F_D = C_d A \, \frac{\rho U^2}{2} \tag{2.39}$$

- C_d is the Drag Coefficient. Since is a coefficient it has not units. It is a term that is going to be explained right bellow.
- ρ is the density $[kg/m^3]$
- U is the relative velocity of the object $[m/s]$
- A is the reference area $[m^2]$

There are various types of drag, some of them are:

- Skin friction drag

  It is caused by the contact between the fluid particles and the body's surface as if it were the friction everybody knows. Its magnitude depends on the body as well as on fluid; it can be decreased if the velocity of the solid increases.

- Form drag

  Also known as pressure drag, it caused by the separation of the fluid upon meeting the solid. Because it is separated, turbulence will be present and a wake is right at the hind face of the body obstructing the motion. That is why is so important to streamline when designing a car, an aircraft, etc .(Skybrary, 2017)

2.8.4 Drag Coefficient

This coefficient is usually determined experimentally. Generally, is constant for almost all bodies over a large range of Re. But once it is determined, this value must be consistent with the referenced area. That is why it is hardly dependent on the geometry (shape) and inclination (National Aeronautics and Space Administration, 2015);except for spheres (because it varies with Re). The analysis for spheres can be carry out from a cylinder since its *2D* dimensional flow is similar from the *3D* flow sphere but the computation of a cylinder is simpler (National Aeronautics and Space Administration, 2014) and also the sphere can be considered smooth.

The drag coefficient measures the opposition applied to an object moving through a fluid.(National Aeronautics and Space Administration, 2015)

$$F_D = \frac{1}{2}\rho A\ U^2 \tag{2.40}$$

$$\begin{aligned} C_D &= \frac{F_D}{\frac{1}{2}\rho U^2 A} \\ C_D &= F_D/qA \end{aligned} \tag{2.41}$$

Where:

- ρ represents the density $[\frac{kg}{m^3}]$
- A represents the area $[m^2]$
- F_D represents the Drag Force $[N]$
- U^2 represents the velocity $[\frac{m}{s}]$
- q represents dynamic pressure $[Pa]$

2.8.5 Flow around a Sphere

Drag on a sphere

In Figure 2.7 it is described a fluid passing through an area where an object with a circular cross section is placed, but the situation changes with the variation of velocity. Each state is described right bellow.

- State 1

  In this case, the flow is low and viscosity is absolutely neglected. There is no boundary layer along the surface and this is what it can be considered as an ideal flow. So, it is completely attached and there is no viscous wake downstream of the cylinder. The flow is symmetric (from upstream and downstream) this means, there is no drag over the object. This is what is called the d'Alambert's paradox. But it is just like an ideal circumstance since in reality, there will be always at least a little amount of viscosity.

- State 2

  This state represents what really happens at low velocities. Stable pair of vortices are formed downside the cylinder. Although the fluid is separated is still steady and the vortices produce high drag on the body.

- State 3

  In this state, the velocity is increased and the vortices formed downstream become unstable, in other words, they have got separated form the sphere; the wake is very wide and a big amount of drag is developed. This is the Karman vortex street. This flow is uniform, steady, but periodic.

- State 4

  For this state, the velocity is bigger; the flow is not periodic anymore but turns into a chaotic wake. On the boundary layer of the windward side of the cylinder, the flow is orderly and laminar while the chaotic wake initiates when the flow turns onto the leeward side of the cylinder. The wake is not as big as in state 3 so drag is slightly smaller.

- State 5

  This state has the highest velocity. The boundary layer converts into a chaotic turbulent flow with vortices of several different scales which come from the a turbulent wake from the body. The separation point is lightly downstream from the laminar separation point, so respect to *State 4*, wake is initially slightly smaller and so the drag. The increase in the velocity, raises turbulent drag to a higher value than the laminar drag value, however, during the transition from laminar to turbulent, the turbulent drag is even smaller than the laminar drag.

The drag for a roughened sphere is smaller than for a smooth sphere, that means than the roughened ball passes to a turbulent flow with a smaller Reynolds number even the two bodies have the same parameters (diameter, velocity and flow conditions) (National Aeronautics and Space Administration, 2015). But for this work, the spheres of the cradle will be considered as smooth.

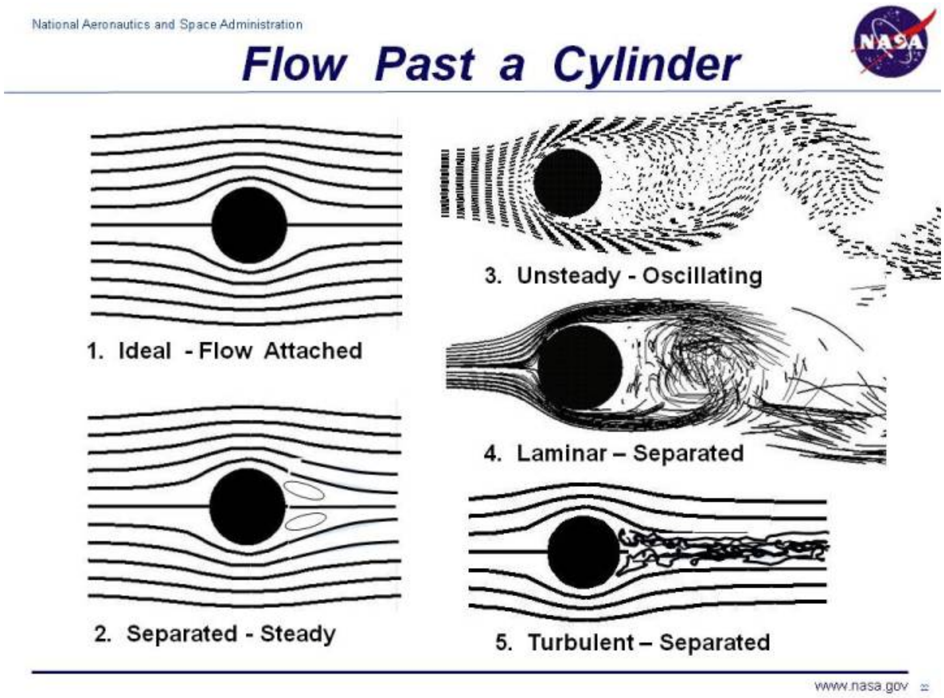


Figure 2.7: Flow passing around a cylinder. In the five cases, the density, viscosity and diameter of the "circle" are the same but the flow velocity is increased, so does the Reynolds number (National Aeronautics and Space Administration, 2015)

Drag on a Sphere in Creeping flow

Creeping flow is a type of flow which considers a low Reynolds number and since that dimensionless number measures the inertial forces over the viscous forces implying that an object moving through this type of fluid will experience slow motion or kind of a "quasi static" state. This, due the high viscosity of the fluid or density, velocity or the length are very small.

Generally, this type of flow exists around microscopic organisms or it can occur where there are small gaps and channels in lubricated bearings or even when an object is deposited in a very viscous fluid (like honey). To simplify the analysis, gravity will be negligible or it will be assumed that it will only affect into the hydrostatic pressure. (Çengel, 2010)

Creeping flow approximation:

$$[Eu]\vec{\nabla}^* P^* \cong \frac{1}{Re}\nabla^{*2}\vec{V}^* \tag{2.42}$$

So, since the Drag force (F_D) of an object with a characteristic length (L) moving through a creeping flow with a certain viscosity (μ) with a velocity (V) can be described as:

$$F_D = constant * \mu V L \tag{2.43}$$

With dimensional analysis, the value of the constant can be obtained because it only depends on the geometry of the body and its orientation in the flow field.

The aerodynamic drag for a *3D* body does not depend on the density but on it speed, length and viscosity.

For a sphere, the constant is 3π, so drag force can be read as:

$$F_D = 3\pi\mu V L \tag{2.44}$$

So, this means that two thirds of the drag are due to viscous forces and the other third, due pressure forces.

Flow Behavior around a Sphere

In the analysis of a heat exchanger, internal and external flow must be considered for the tubes and also, for many sports like soccer, tennis or golf, is very important to consider that the flow that goes around the balls (spherical). Thus, for this work it is also relevant to understand how the fluid influences into the behavior of the cradle.

One vital factor for the analysis is logically the Reynolds number but for its computation, the characteristic length will become the diameter and the fluid velocity will turn more uniform as it comes close to the bodies.

As shown in *State 2* (Figure 2.7), when fluid enters in contact with the body, some patterns are formed. It seemed like the fluid splits in two forming something similar to two arms enclosing the sphere. The stagnation point is beat by the midplane fluid particles and that is where the fluid stops and the pressure increases while in the flow direction, the pressure is reduced and consequently, the velocity grows.

When there are low upstream velocities or said in another way, when the Reynolds number is smaller that *1*, the fluid contours completely the body if the velocities are high, the fluid still holds the body but now, periodic vortexes are formed just like pressure but much lower than the pressure at the stagnation point.

It is possible to have an idea of how the fluid can behave with the drag coefficient (C_D) but friction and pressure drag are relevant too.

Since there is high pressure near the stagnation point and low pressure at the other side of the sphere, a net force is produced over the body on the direction of the flow. At low Reynolds number, ($Re \lesssim 10$) the drag force is mainly due to friction drag.

As seen in Figure 2.8, if $Re \lesssim 1$, the flow is a creeping flow; decreasing the drag coefficient with the increase of the Reynolds number. For a sphere it is: $C_D = \frac{24}{Re}$[III]

[III]This relation will be explained further

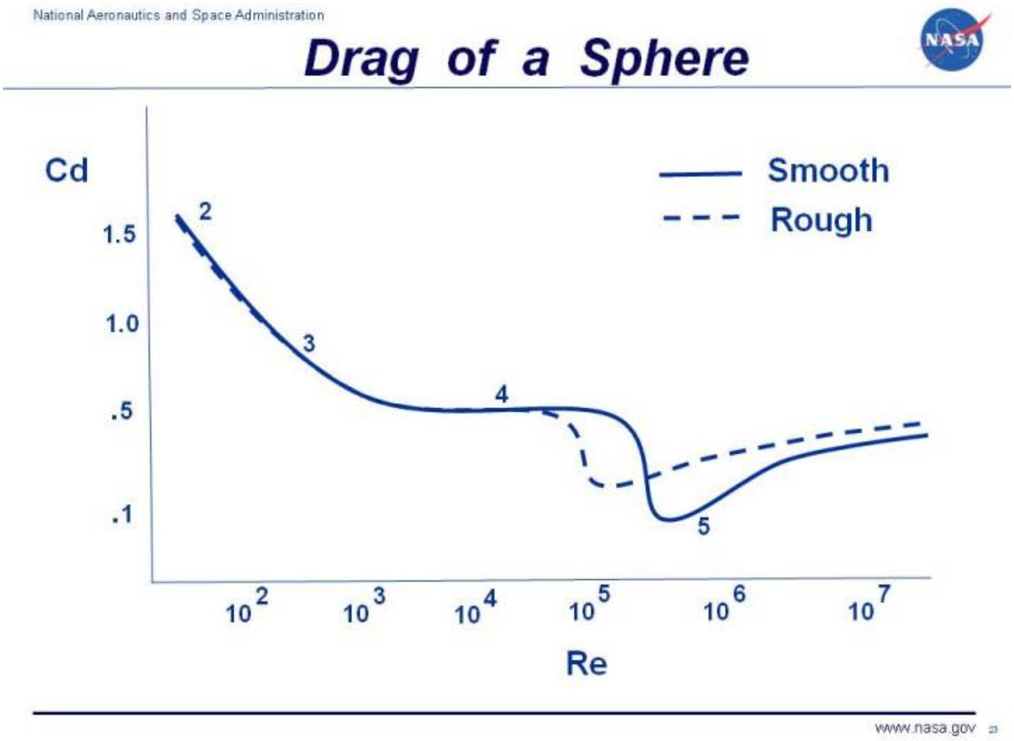


Figure 2.8: Experimental results to obtain the Drag Coefficient versus Reynolds number (National Aeronautics and Space Administration, 2015).

To understand why C_D is equal to $\frac{24}{Re}$, it is better employing spherical coordinates for the calculations since the bodies used in this work are spheres. To get the drag force, the stresses over the sphere must be integrated (the normal and tangential ones). And since stress is equal to force per unit of area:

$$
\begin{aligned}
F_D &= \hat{e}_z \int_A \mathcal{T} \cdot \hat{n}\, dA \\
\mathcal{T} \cdot \hat{n} &= -p\hat{e}_r + \tau_{r\theta}\hat{e}_\theta = [-p\sin(\theta) + \tau_{r\theta}\cos(\theta)]\hat{e}_y - [p\cos(\theta) + \tau_{r\theta}\sin(\theta)]\hat{e}_z \\
F_D &= \int_0^{2\pi} d\phi \int_0^{\pi} [p\cos(\theta) + \tau_{r\theta}\sin(\theta)]\sin(\theta)\, d\theta \\
&= -2\pi R^2 p_\infty \int_0^{\pi} \cos(\theta)\sin(\theta)\, d\theta - \frac{3\mu W_0}{2R}\int_0^{\pi} \sin(\theta)(\cos(\theta))^2\, d\theta - \frac{3\mu W_0}{2R}\int_0^{\pi} (\sin(\theta))^3\, d\theta \\
&= -2\pi R^2 p_\infty(0) - \frac{3\mu W_0}{2R}(\frac{2}{3}) - \frac{3\mu W_0}{2R}(\frac{4}{3}) \\
&= -2\pi R^2 - \frac{3\mu W_0}{R} \\
&= 6\pi R\mu W_0
\end{aligned}
\tag{2.45}
$$

And since

$$
\begin{aligned}
C_D &= \frac{F_D}{\frac{\rho}{2}\pi R^2 W_0^2} \\
&= \frac{12\mu}{\rho W_0 R} \\
&= \frac{24\mu}{\rho W_0 D}
\end{aligned}
\tag{2.46}
$$

And because

$$
\begin{aligned}
Re_D &= \frac{\rho W_0 D}{\mu} \\
C_D &= \frac{24}{Re_D}
\end{aligned}
\tag{2.47}
$$

Where:

- μ is the viscosity of the fluid
- W_0 is the flow in the system
- R is the radio of the sphere
- ρ is the density of the fluid
- D is the diameter of the sphere

If $Re \cong 10$, a separation of the flow will start in the hinder part of the body
If $Re \cong 90$, the flow separation is by means of vortex shedding
If $Re \cong 10^3$ the separation is higher and drag is mostly due to pressure drag

Drag coefficient continues decreasing if Re increases but a decrease in drag coefficient does not mean a decreases in drag. And since drag is proportional to the square of velocity, if velocity grows at a high Re, the drag coefficient goes down.

This has sense since drag is a force of resistance.

2.9 Hertzian Contact

When two solid bodies touch each other, there are two kinds of contact which can be recognized according to the kind of contact: Conforming and non conforming contact. A conforming contact consists in the perfect or close to perfect "match" of two surfaces of two solid bodies without any deformation.

Heinrich Hertz in 1881, was the first one who disclosed this kind of analysis (Williams, Dwyer-Joyce, 2000). The analysis is ideal for parabolic surfaces (therefore, the pressure has a parabolic distribution) but it can be also useful for spheres, cylinders and ellipsoids. But the analysis has some conditions that must be fulfilled:

- The surfaces of the bodies must be continuous, smooth, nonconforming and frictionless
- The size of the contact area must be small compared with the size of the bodies (in this case, of the spheres), so the forces that may cause deformations must also be small
- Near the contact zone, it can be considered that the solid will behave as an elastic half-space.[IV]
- The gap between the undeformed surfaces can be approximated to (using Cartesian coordinates):

$$h = Ax^2 + By^2 \tag{2.48}$$

Where x and y are the orthogonal coordinates that are located in the common tangent plane to the two surfaces.

This concept can help to know or to approximate the shape of the area of contact and how it changes with the variation of the load or loads; how the surface behaves (tractions), as well as, if the traction is distributed in a normal or even tangential way.

If the analysis is at the micro scale, small surfaces irregularities will be disregarded because these could chase to a discontinuous contact or to a quite significant contact pressure variation. If the analysis is at the macro scale, the profiles of the surfaces are continuous up to their second derivative in the contact region.

[IV] In the next subsection, it will be explained what is a Perfect Elastic Collision

2.9.1 Contact of Spheres

In a Newton's Cradle, the balls collide, which means that there is going to be always contact between at least, two balls. Due the structure of the spheres, they touch themselves via one single point, and so, the contact will be non-conforming. But no matter it is a "small point", even that point has an area which is the area of contact (circular area of contact); certainly, this area will always be much smaller compared with the size of the system (in this case, the size of the balls). Also, the stresses will always get concentrated near contact zone but they are also not going to be very influenced by the shape of the shape of the body.

Considering two spheres with radios: R_1 and R_2. The spheres now have contact. The point of contact will be the origin of the coordinate system. Employing cylindrical coordinates, near from the origin where $r << R_1$ and $r << R_2$ (r is the radial position), the area of contact will be:

$$z_1(r) = -\frac{r^2}{2R_1} \qquad (2.49)$$

$$z_2(r) = \frac{r^2}{2R_2} \qquad (2.50)$$

If the radial position increases, so does the vertical position. And also, a gap between the spheres will start to appear. The gap begins right just where the point of contact finishes.

If the spheres have the same geometrical characteristics and the radial position is the same, the whole gap will be the sum of the vertical positions with respect to the radial position. It can be expressed like:

$$\begin{aligned}\triangle^{\circ}(r) &= z_2(r) - z_1(r)\\ \triangle^{\circ}(r) &= \frac{r^2}{2R_2} + \frac{r^2}{2R_1}\\ \triangle^{\circ}(r) &= \frac{2R_2r^2 + 2R_1r^2}{4R_1R_2}\\ \triangle^{\circ}(r) &= \frac{r^2(R_2 + R_1)}{2R_1R_2}\end{aligned} \tag{2.51}$$

If the bodies are submitted to a force (F) they can start to deform or an indentation could be produced, the contact will be now an area, a circular area defined by:

$$\triangle\,(r) = \triangle^{\circ}(r) - \varsigma + [w_2(r) + w_1(r)] \tag{2.52}$$

Where:

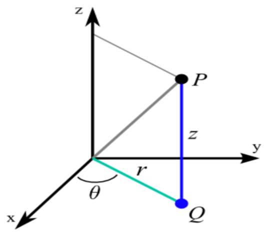
- ς is the indentation and it can be the maximum overlapping of the spheres in case the bodies do not deform.
- $w_2(r)$ and $w_1(r)$ are the local displacements with respect to the radial position

The analysis of Hertz was solved with the assumption that $\triangle(r) = 0$

In the case the bodies in contact are identical, the two radius are also equal and substituting in Equation 2.51 the relation between the ratios will be:

$$\begin{aligned}R_1 = R_2 &= R\\ \frac{2R}{2R^2} &= \frac{1}{R}\end{aligned} \tag{2.53}$$

Figures 2.9 and 2.10, displays a graphic representation of the described right above.



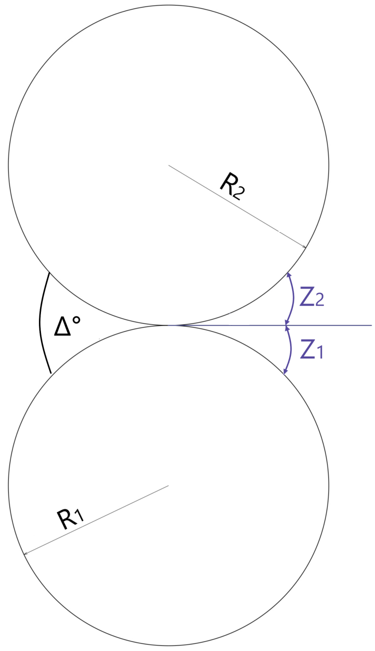


Figure 2.9: Diagram of contact between two identical spheres.

Some material properties must be considered, the ones that matter when the spheres enter in collision, are the **elastic collisions** and if the bodies are identical, the contact modulus can be defined as:

$$k_1 = \frac{1-\nu_1^2}{E_1} \qquad k_2 = \frac{1-\nu_2^2}{E_2} \tag{2.54}$$

$$k_1 = k_2 \qquad k = \frac{1-\nu^2}{E} \tag{2.55}$$

A perfect elastic collisions is a type of collision where there is no kinetic energy nor momentum loss. Suppose two identical spheres (just like the ones sued for the project), traveling with the same speed ($v_{1i} = v_{2i}$) but at opposite ways; at some point, they collide; if they bounce off each other, without losing their original

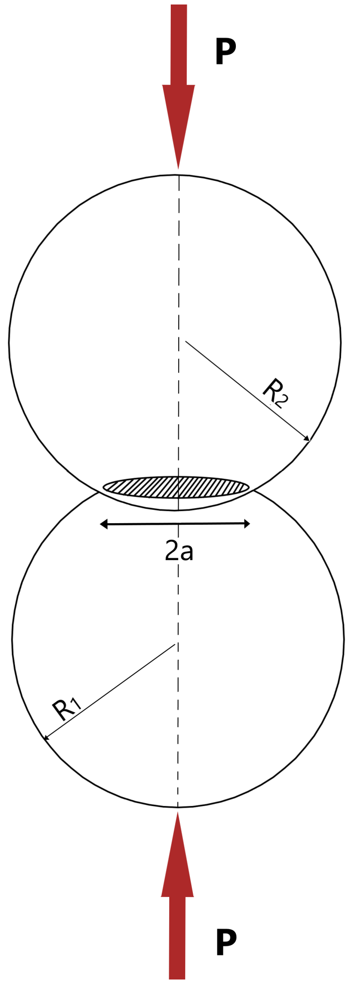


Figure 2.10: Diagram of two identical spheres being submitted under a pressure force.

speed ($v_{1f} = v_{2f}$), perfect elastic collision is generated. Initial velocities are not required to be equal).

The gap can be redefined as:

$$\triangle^{\circ}(r) = \frac{r^2}{R} \tag{2.56}$$

So, under the contact area (which starts to grow whereas the force increases), there will be a parabolic pressure load distribution (Tribonet, 2017) defined as:

$$q(r) = \sigma_z(r,0) = \sigma_{z0}\sqrt{1 - (\frac{r}{a})^2} \tag{2.57}$$

Where σ_{z0} is the normal stress at the center ($r = 0$) and a is circular contact radius:

$$\begin{aligned} a &= \frac{\pi R k \sigma_{z0}}{2} \\ a &= (\frac{3RkF}{4})^{\frac{1}{3}} \end{aligned} \tag{2.58}$$

And so, F will be:

$$F = \int_0^{2\pi}\int_0^{a} q(r)\, rdr\, d\theta = 2\pi\sigma_{z0}\int_0^a r\sqrt{1-(\frac{r}{a})^2}\, dr = \frac{2\pi a^2\sigma_{z0}}{3} \tag{2.59}$$

$$F = \frac{2\pi a^2 \sigma_{z0}}{3} \tag{2.60}$$

Which means that, the higher the radius a and the normal stress σ_{z0}, the higher the applied force towards the spheres due the increases in the contact area. And so, the indentation can be read as:

$$\begin{aligned} \varsigma &= \pi a k \sigma_{z0} \\ \varsigma &= \frac{2a^2}{R} \\ \varsigma &= (\frac{9k^2F^2}{2R})^{\frac{1}{3}} \end{aligned} \tag{2.61}$$

Meaning that the higher the force and deformation, the higher the indentation but it decreases if the radius grows.

And finally, the normal stress (maximum contact pressure) can be redefined as:

$$\sigma_{z0} = \frac{1}{\pi}(\frac{6F}{R^2k^2})^{\frac{1}{3}} \tag{2.62}$$

Or also as:

$$\sigma_{z0} = \frac{3F}{2\pi a^2} \tag{2.63}$$

σ_0 is also known as the Hertz stress (Tylor, 2016). Under that load, the centers of the spheres move together, certainly with a displacement; that is why also the gap between the bodies, changes. As it can be seen in Figure 2.11

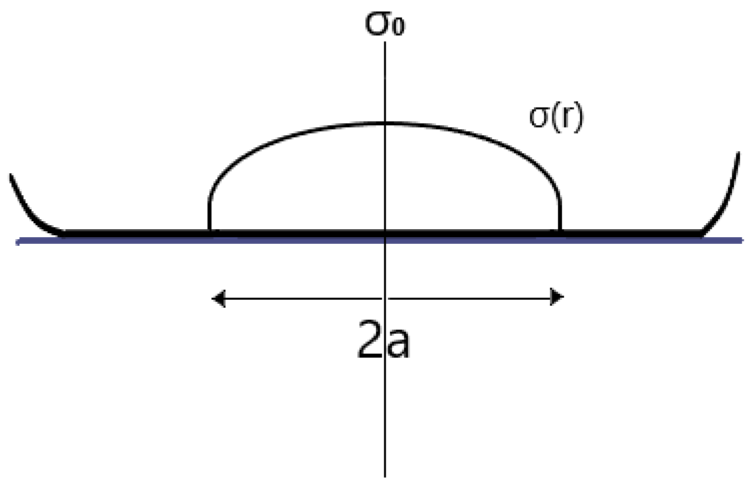


Figure 2.11: Diagram of the normal stress caused by the applying of a pressure force and the area of contact.

2.10 Damping

To start analyzing the behavior of a system, as already told above, it is necessary to define the structure and the forces acting on it. The outline of the motion is described through a differential equation and the solution through a mathematical model.

Theoretically, it is assumed that in case of a system with free vibrations, the amplitude of the signals and the velocity are invariable; or, in other words, they would remain constant, but in the practice, this not happens, because as time passes, the amplitude decreases, so the vibrations are damped out (Timonshenko, 1937). That is the reason why, at the moment of doing an analysis, the damping forces are critical. Damping forces are the result of, for example, friction of a dry or lubricated sliding, air or fluid resistance, electrical damping, internal friction, etc. In case of a system with air as the fluid, the resistance is proportional to the square of velocity.

2.10.1 Free Vibration with Viscous Damping and Logarithmic Decrement

Considering that a vibrating body is at motion but with a certain resistance proportional to the velocity, the equation of motion can be defined as:

$$\frac{W}{g}\ddot{x} = W - (W + kx) - c\dot{x} \tag{2.64}$$

Where the term $c\dot{x}$ refers to the damping force which is proportional to the velocity ($\dot{x}$) and the term has a negative sign because it acts in the opposite way to velocity. The coefficient c represents the magnitude of the damping force when the velocity is equal to unity and W represents the weight of force.

Dividing Equation 2.64 by $\frac{W}{g}$, and using $\omega_0^2 = \frac{kg}{W}$ and $\frac{cg}{W} = 2h$. (Knowing that $\omega_0 = \sqrt{\frac{k}{m}}$ [V] and $h = \frac{c}{2m}$ [VI], the next equation is obtained:

$$\ddot{x} + 2h\dot{x} + \omega_0^2 x = 0 \tag{2.65}$$

Since is a differential equation of the form $\ddot{x} + \omega_0^2 t = 0$, the solution would be:

$$x = Ae^{rt} \tag{2.66}$$

Where A and r are unknown constants. (Majewski, 2016)

And substituting Equation 2.66 in 2.65, it is obtained:

$$r^2 + 2hr + \omega_0^2 = 0$$

$$\tag{2.67}$$

And so:

$$r = -h \pm \sqrt{h^2 - \omega_0^2}$$

So there are two possible results:

$$r_1 = h + \omega_0 i \qquad r_2 = -h - \omega_d i$$

[V] To recap the explanation of this equation, see Equations 2.3 and 2.4

[VI] To revise the explanation of this equation, check from Equations 2.76 to 2.80

Substituting the roots in the solution (Equation 2.66), the next particular solutions are obtained.

$$x_1 = \frac{C_1}{2}(e^{r_1 t} + e^{r_2 t}) = C_1 e^{-ht}\cos\omega_d t$$
$$x_2 = \frac{C_2}{2i}(e^{r_1 t} - e^{r_2 t}) = C_2 e^{-ht}\sin\omega_d t$$

Moreover, by adding or subtracting of these solutions, the imaginary unit can be removed.

$$x = e^{-ht}(C_1\cos\omega_d t + C_2\sin\omega_d t) \tag{2.68}$$

ω_d refers to damped angular frequency since the motion can be classified as Damped Harmonic Motion, so it can be understood that the angular frequency will change along time ($\omega_d \neq \omega_0$), that is to say that it turns a little smaller, that is, a decrement. ω_d is given by:

$$\begin{aligned} \omega_d &= \sqrt{\omega_0 - h^2} \\ \omega_d &= \sqrt{k/m - c^2/4m^2} \end{aligned} \tag{2.69}$$

This expression points that the bigger the coefficient c, the faster the amplitude will decay. This new frequency turns zero when c has a certain magnitude that:$\frac{k}{m} - \frac{c^2}{4m^2} = 0$. In the following section (see Equation 2.76 and 2.77), it is explained the justification of the damping coefficient.

With all the information above, the period can be determined:

$$\tau = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_0}\frac{1}{\sqrt{1 - \frac{h^2}{\omega_0}}} \tag{2.70}$$

With this expression, it can be stated that with damping, the period, along time, will increase, but if the term h is small compared with ω_0, this increasing is minute, so it can be concluded that small viscous damping does not really affect the period of vibration.

To determine C_1 and C_2, firstly, it is needed to assume that at $t = 0$, the body is displaced from the equilibrium by a certain x_0 distance and has an initial velocity $\dot{x_0}$. With those initial conditions, $x_0 = C_1$ and differentiating the expression with respect to time for $t = 0$, it is obtained: $C_2 = \frac{(\dot{x}+hx_0)}{\omega_d}$and through substitution with Equation 2.68, the next expression is obtained:

$$x = e^{-ht}(x_0 \cos \omega_d t + \frac{\dot{x} + hx_0}{\omega_d} \sin \omega_d t) \tag{2.71}$$

The term $\cos \omega_0 t$ depends on the initial displacement (x_0) and the second term, which is proportional to $\sin \omega_0 t$ depends on the initial displacement, but also on the initial velocity.

In Figure 2.12, the terms of the equation are represented. The curve is tangent to the function $x = A_0 e^{-ht}$ at the points n_1, n_2andn_3 when $t = 0,\ t = \tau\ and\ t = 2\tau$. With the function $x = -A_0 e^{-ht}$ at points $n_1, n_2\ and\ n_3$ when $t = \frac{\tau}{2}, t = \frac{3\tau}{2}$, but those points do not agree with the points of the extreme displacements from the beginning, which means there is **damping**. Notice that the interval of time needed to displace from the middle point to one of the extreme position, as seen in Figure 2.12 diminishes as time passes because the extreme positions changes making the body to move within a shorter distance.

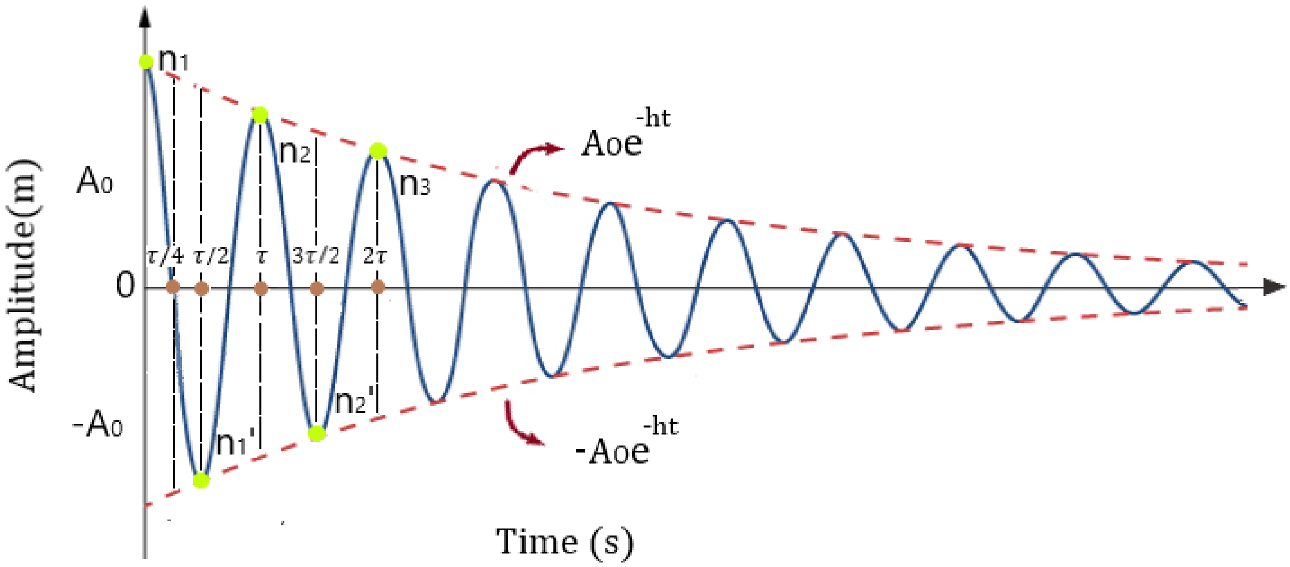


Figure 2.12: Logarithmic decrement of a signal with the function $x = Ae^{rt}$.

The relation between these changes in the limits through time, depends on the magnitude of h, as seen Equation 2.64. Because as seen in the general solution and in Figure 2.12, the amplitude will decrease with a ratio of $e^{-hr} : 1$. Which means that it satisfies the geometric progression, Logarithmic decrement as proven in Figure 3.6. In case of having already the data of the length of the periods against time, the only thing needed is to to figure out the ratio for the amplitude of vibration after certain cycles. Considering this analysis is about the logarithmic decrement, the ratio will be the difference between two consecutive amplitudes measured in two consecutive periods (t and $t + \tau$) For example:

$$n_\tau = \frac{2\pi}{\omega_0} \frac{h}{\sqrt{1 - \frac{h^2}{\omega_0^2}}} \tag{2.72}$$

The logarithmic decrement can be understood as the natural logarithm of a ratio of two consecutive amplitudes.

To have a visual overview of how this phenomenon occurs, a vector can be represented. In Figure 2.13 the vector $\overline{OA}$ has a magnitude of $x_0 e^{-ht}$ rotating with a constant angular velocity ω_0.Measuring the angle of rotation in clockwise direction from the x-axis, the $\overline{OA}$ projection which is OA_1 is equal to $x_0 e^{-ht}\cos\omega_d t$ which is the representation of the first term of Equation 2.71. Occupying now a vector $\overline{OB}$ perpendicular to $\overline{OA}$ and with magnitude $e^{-ht}\frac{(\dot{x_0}+hx_0)}{\omega_d}$ in order to project it on the x-axis, the second term of the Equation 2.71 is accomplished. The result of addition of these two vectors, is the new vector $\overline{OC}$, which projection is the total expression of Equation 2.71. Its magnitude is equal to: $\overline{OC} = e^{-ht}\sqrt{x_0^2 + \frac{(\dot{x}+hx_0)^2}{\omega_d^2}}$.

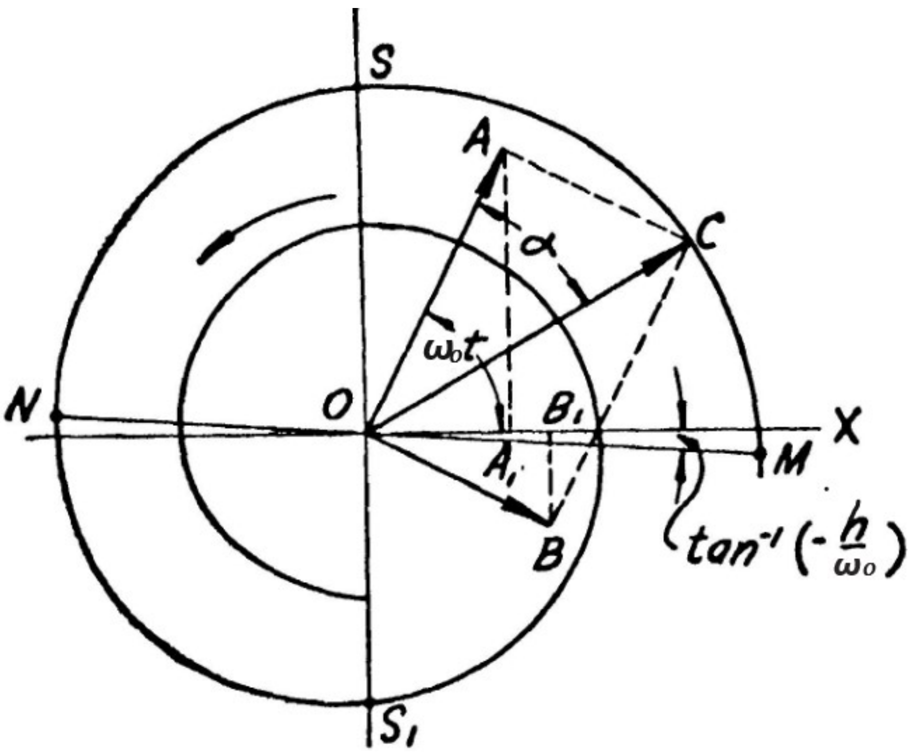


Figure 2.13: Illustration of the trajectory of a rotational body (Timonshenko, 1937).

The angle formed with x-axis is $\omega_d t - \alpha$ where $\alpha = \arctan\frac{\dot{x_0}+hx_0}{\omega_d x_0}$. So, at the end, the expression of Equation 2.71 can be interpreted as:
$x = e^{-ht}\sqrt{x_0^2 + \frac{(\dot{x}+hx_0)^2}{\omega_d^2}}\cos(\omega_d t - \alpha)$.

As $\overline{OC}$ rotates, the point C traces a logarithmic spiral which tangent creates a constant angle equal to $\arctan(\frac{-h}{\omega_d})$ with the perpendicular to the radius $\overline{OC}$. The vertical tangents of the spiral represent the extreme positions. The points where the spiral and the vertical axis meet, indicate that the body is passing through the equilibrium position. And evidently the time required from extreme positions to the subsequent others, is shorter. In Figure 2.13 the time for the angle SON is lower than the subsequent which is angle NOS_1. However, the time between two successive extreme positions (In the Figure, points M and N), is the same, which represents a half of a period.

In this part it was discussed Equation 2.65 assuming that $\omega_0^2 > h^2$. On the contrary, $\omega_0^2 < h^2$, the roots become real and negative and substituting in Eq 2.66, the general solution would be:

$$x = C_1 e^{r_1 t} + C_2 e^{r_2 t} \tag{2.73}$$

This means a very large viscous resistance so the body does not vibrate but only creeps gradually back towards that position. The critical value of damping is only valid if $h = \omega_0 =$ with:

$$\begin{aligned} c_{cr} &= 2\sqrt{\frac{kW}{g}} \\ c_{cr} &= 2\sqrt{km} \end{aligned} \tag{2.74}$$

But knowing that h can be also expressed as $\frac{c}{2m}$[VII], the critical damping coefficient is also understand as:

$$\frac{c_{cr}}{2m} = \omega_0$$
$$c_{cr} = 2m\sqrt{\frac{k}{m}} \tag{2.75}$$

So it can also be said that $r_1 = r_2 = -\frac{c_{cr}}{2m} = -\omega_0$

This kind of motion causes the quickest approach to a zero amplitude.

2.10.2 Damping Coefficient

The damping coefficient is a mathematical expression to regard the energy loss in a vibrating system (Stutts, 2018).Since this term acts affects the system as if it there friction, the system can be defined as: $F = c\nu$. Where c refers to the coefficient and ν to the velocity. Its units are *force/velocity*$=(\frac{N}{\frac{m}{s}} = \frac{kg}{s})$.

From Equation 2.69, c can be determined:

$$c^2 = \frac{k}{m}\,4m^2 = 4km \tag{2.76}$$

$$c = \sqrt{4km} = 2\sqrt{km} \tag{2.77}$$

From Equation 2.69 in can be determined that:

$$h = \frac{c}{2m} \tag{2.78}$$

[VII]As mentioned above, the explanation would be found un the next subsection starting from Equation 2.77

And so, to prove the balance of the expression, Equation 2.76 is substituted into 2.69:

$$\frac{k}{m} - (\frac{c}{2m})^2 = \frac{k}{m} - \frac{c^2}{4m^2} \qquad (2.79) \qquad = \frac{k}{m} - \frac{4km}{4m^2} = \frac{k}{m} - \frac{k}{m} = 0 \qquad (2.80)$$

A damping coefficient of 100% would mean a critical damping manifesting as a not freely vibration. A coefficient of 1% would mean that the amplitude may decay approximately 6% over one period of oscillation. (CreoParametric, no date)

2.10.3 Damping ratio

This is a dimensionless parameter which describes how an oscillating vibrating body comes or comes back to rest. In other words, this ratio can tell how rapidly the amplitude decays against time. If there were not damping, the body would never rest.

$$\epsilon = \frac{actual\ damping\ coefficient}{critical\ damping\ coefficient}$$
$$\epsilon = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{mk}} \qquad (2.81)$$

But, for this work

$$\epsilon = \frac{c}{2\sqrt{gL}}$$

- If = 0 the motion turns into a undamped motion.
- If < 1 the motion turns into a underdamped motion.

  If the system is excited by a force, it would eventually oscillate but gradually come into rest.
- If = 1 the motion turns into a critical motion.

Even the system is displaced from its equilibrium position, it does not overshoot that position immediately and with this coefficient, the rest is reached at a minimum time.

- If > 1 the motion turns into a overdamped motion.
  The system does not vibrate. (School of Engineering Brown University¸no date)

Now, the effect that damping has in the pendulum experiment would be shown and eventually, the concept of damping would be more clear.

With the information above, Equation 2.65 can be also read as:

$$\ddot{x} + 2\epsilon\omega_n\dot{x} + \omega_n^2 x = \frac{F(t)}{m} \tag{2.82}$$

Over-damped motion

This motion occurs with the next condition: If $h > \omega_0$ and if $\epsilon = \frac{c}{c_{cr}} > 1$

This conditions cause the roots to be real but negative with this solution and since for this specific form of motion $h^2 - 4\omega_0^2 > 0$ $x(t) = A_1 e^{\frac{1}{2}(-h-\sqrt{h^2-4\omega_0^2})} + A_2 e^{\frac{1}{2}(-h+\sqrt{h^2-4\omega_0^2})}$. (Wolfram, 2021)

With the name of this motion it can be understood that, its behavior would be an exponential decrease against time making the approach to zero amplitude to be slower. Its damping ratio can be read as: $\epsilon = \frac{c}{c_{cr}} = \frac{c}{2m\omega_0}$. This damping coefficient is always greater than the undamped resonant frequency.

Underdamped motion

This motion occurs with the next condition: If $h < \omega_0$ and if $\epsilon < 1$. This makes the approach to a zero amplitude to be quicker than the critical damping but unlike critical damping, it oscillates around it (zero amplitude). (Hyperphysics, no date)

This conditions cause the roots to be imaginary and can be read as: $r_{1,2} = -h \mp i\sqrt{\omega_0^2 - h^2}$. So, its solution would be: $x(t) = e^{-ht}(C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)) = a_0 e^{-ht} \sin(\omega_d t + \psi)$. Where $\omega_d = \sqrt{\omega_0^2 - h^2}$ which represents the natural frequency with damping, the initial amplitude, $a_0 = \sqrt{C_1^2 + C_2^2}$ and the phase angle $tan\psi = \frac{C_1}{C_2}$.

For a sub-damped motion the damping coefficient can be presented as: $c = \frac{2m}{T_d \ln(\frac{a_1}{a_2})}$ and so $T_d \cong T_0 = 2\pi\sqrt{km}$ as well as you with $c = \frac{\sqrt{km}}{\pi} \ln(\frac{a_1}{a_2})$.

The frequency of the damped vibrations (ω_d) is always smaller than the natural frequency (ω_0). So it can be assumed that $\epsilon \ll 1$ and also that $\omega_d \approx \omega_0$.

Chapter 3

Results of the Experiments and Simulations

3.1 Physical Experiments

In this section, the procedure and the devices of the experimentation will be explained.

The performance consisted on the evaluation of the system behavior with a specific initial amplitude. The displacement of the sphere was studied in order to get the damping ratio and what it represents for this specific experiment.

3.1.1 Safety

- Laboratory coat
- Safety glasses
- Long pants
- Safety footwear
- Enclosed illuminated area
- Record in a safe place to safeguard the camera and equipment
- Isolate as far as possible the work area from external motion, noise and disruptions.

3.1.2 Materials

- Oscillating system (Newton's Cradle)[I]
- Smartphone camera
- Clean and smooth surface
- Enough natural or fulgent light
- Ruler
- Level Ruler (optional)
- PC device with *Matlab*

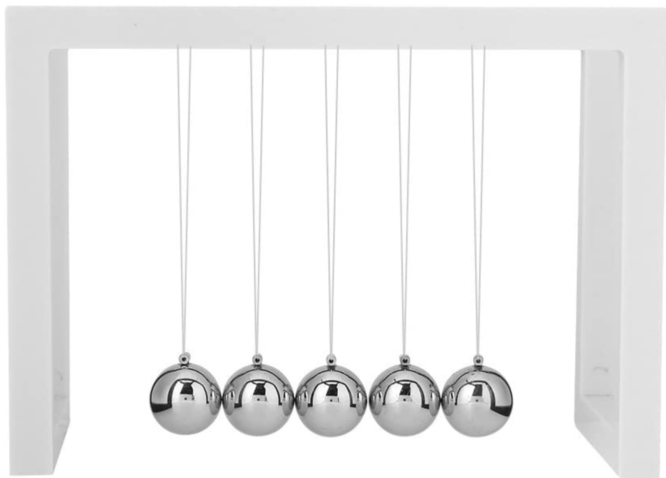


Figure 3.1: Cradle acquired for the experiments (Amazon.com,Inc, 2018).

3.1.3 Experiment Description

The experiment consisted of recording the behavior of an oscillating sphere acting as a pendulum and evaluating its data in order to compute and therefore, analyze the displacement the sphere had as well as its fundamental and damped frequency, its damped natural frequency and finally, with that information, its damping ratio can be obtained. In Figure 3.2, the material and the procedure the experiments were performed are presented.

[I] The device the experiments were performed with, was acquired from *Amazon.* The cradle was sold by *Amoq*, a Chinese distributor. In Figure 3.1, the device employed for the experiments is shown.

Figure 3.2: Necessary material for the experiments.

3.1.4 Device description

The main dimensions of the device are explained in the table bellow. In Figure 3.3 the parameters stated in Table 3.1 are marked.

Dimension	Frame	Strings	Spheres
Frame length (L_f)	16cm=6.30 in	-	-
String length (L_s)		9.3cm= 3.66in	
Height (H)	11.6 cm=4.567 in	-	-
Width (W)	8.7cm = 3.43 in	-	-
Thickness (S)	1cm = 0.04 in	-	-
Spacing between each pair of strings (b)	-	1.8cm= 0.7087in	-
Diameter (D)	-	-	1.8cm= 0.7087in
Mass of the spheres (m)	-	-	24g=0.024kg

Table 3.1: Principal dimensions of the Newton's Cradle.

- Frame: The frame is made of plastic and to prevent the frame from slip, a sponge mat is stuck at the bottom part of the frame.
- Strings: The cradle has double strings attached to each sphere. They are made of nylon.
- Spheres: The device has five steel smooth spheres.

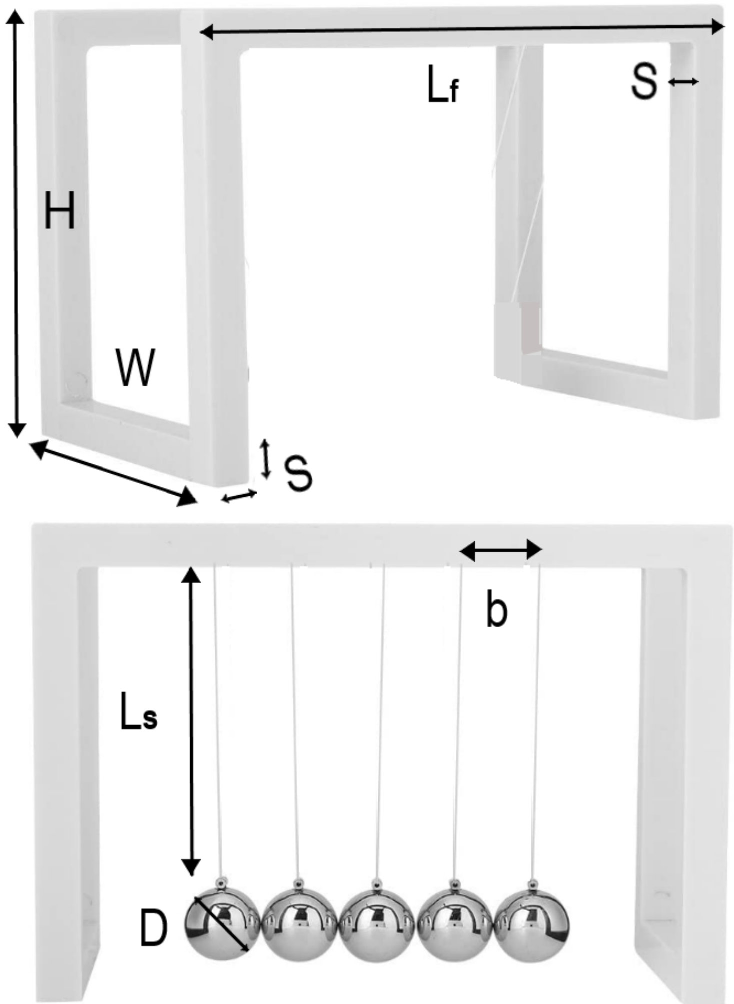


Figure 3.3: Diagram with the cradle dimensions (Amazon.com,Inc, 2018).

3.1.5 Procedure for the Experiment Performance

The illustration of the steps followed for the experimentation is presented in Figure 3.4

- Check the device is in proper conditions:

  - Strings stretched, well-aligned and untangled

  - Well-cleaned spheres

  - The frame properly leveled with the surface

- Check that the surface where the device is going to be placed, is smooth, otherwise, the results can be altered.

- Place the camera on the surface and confirm the lens of the camera are centered and aligned with the midpoint of the ball.

- Record the motion with enough light as well with the highest possible quality.

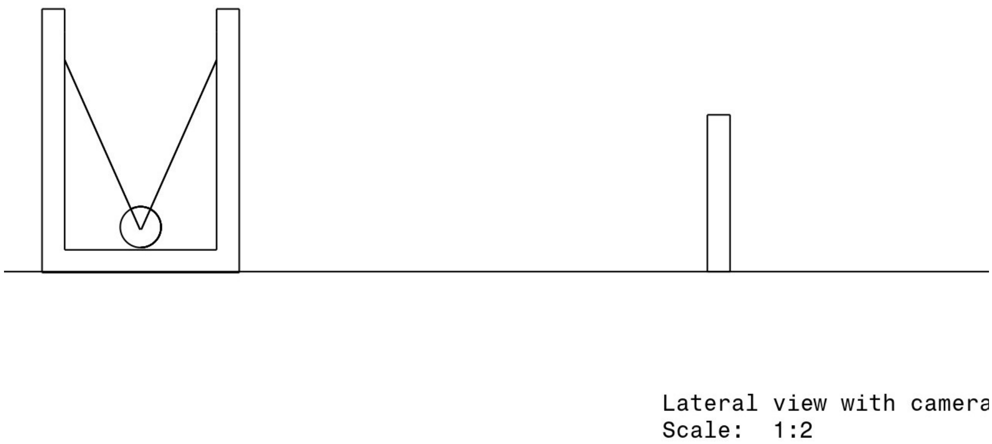


Figure 3.4: Lateral view of the experimental procedure with the device at the left side and the camera at the right side; both placed in the smooth surface.

3.1.6 Image Analysis of the Physical Experiments

The camera had a distance of approximately $40cm$ from the device. The video recorded a amplitude pendulum motion of $\frac{\pi}{6} = 30°$. The video data was read with the software *Matlab*.The video lasted a little more than 20 seconds and since its quality was of 30fps (frames per second), the whole information was obtained from 600 frames.

A black cardboard worked as the background because to analyze the data with *Image Processing Toolbox*[II] of *Matlab* it is necessary to convert the frames into binary images (Mathworks, no date).This whole process is further explained right bellow.

The simulation consisted in following the motion of the centroid of the sphere. Thus, the frames had to be scaled to pixels.

Considering that the frontal view of a sphere is a circle, a disk-shaped structuring element was generated so at every frame, this shape showed up (Mathworks, no date).Afterward, the silhouette was improved with other operations such as:

- Deleting pieces that were not the ball (Mathworks, no date)
- Refining its contour (Mathworks, no date)
- Morphological closure (Mathworks, no date)

After the data was acquired by reading the video and with the previous operations already mentioned, a "exchange" of colors was declared; for example, since the frame was white, in the simulation it was turned into black color. So on each frame the "disk" was identified with gradient computation. Therefore, when the disk was oscillating, it was detected, and the matrix was filled with a *1* whereas the rest of the region (not at motion) filled the matrix with a *zero (0)*. (Mathworks, no date)

[II] *Image Processing* is a method to perform some operations or treatments to an image, in order to extract some useful information from it.

In the meantime, the centroids were collected to plot a displacement against time graph. And then, the peaks were found in order to determine the *Logarithmic Decrement* which can be understood as the natural logarithm of the ratio of the amplitude between two cycles (Universidad Politécnica de Madrid, no date).In other words, it shows the reduction of the amplitude in the immediately following cycle. The maximum amplitude (peak) of the previous cycle is always larger than the next one. Due the Logarithmic Decrement Method, provides a damping ratio and a fundamental frequency every two peaks, an histogram was constructed to get the main damping ratio and a main fundamental frequency. And finally, being able to compute the damped frequency of the system. The demonstration of this procedure is showed in the next section.

3.1.7 Results of the Image Processing

The results in the simulation are displayed in Table 3.2 and also shown in Figures 3.5, 3.6 and 3.7

Damping ratio	Damped frequency	Fundamental frequency
0.00169	$10.6866Hz$	$1.70082Hz$

Table 3.2: Numerical results.

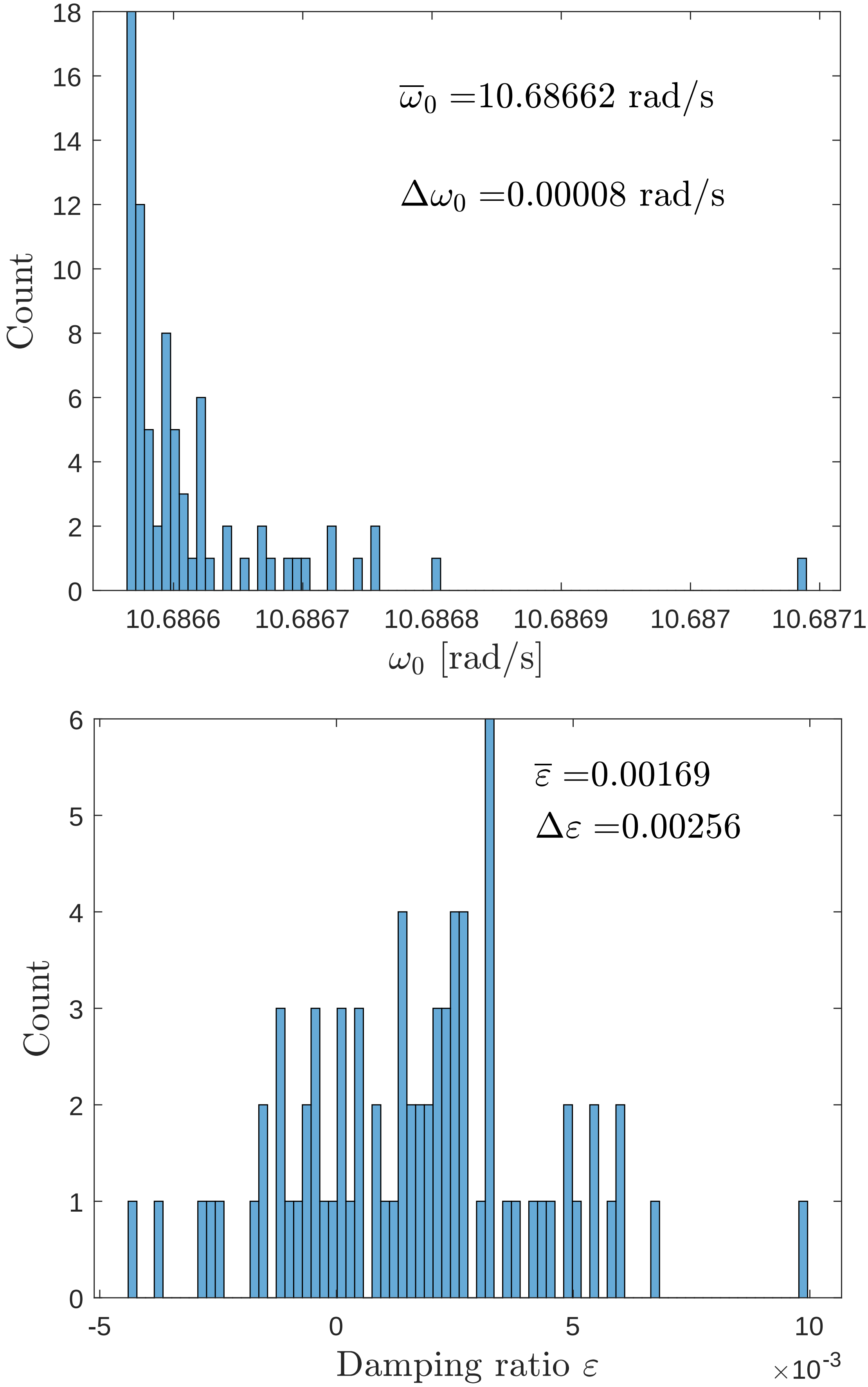


Figure 3.5: Natural angular frequency and Average Damping ratio.

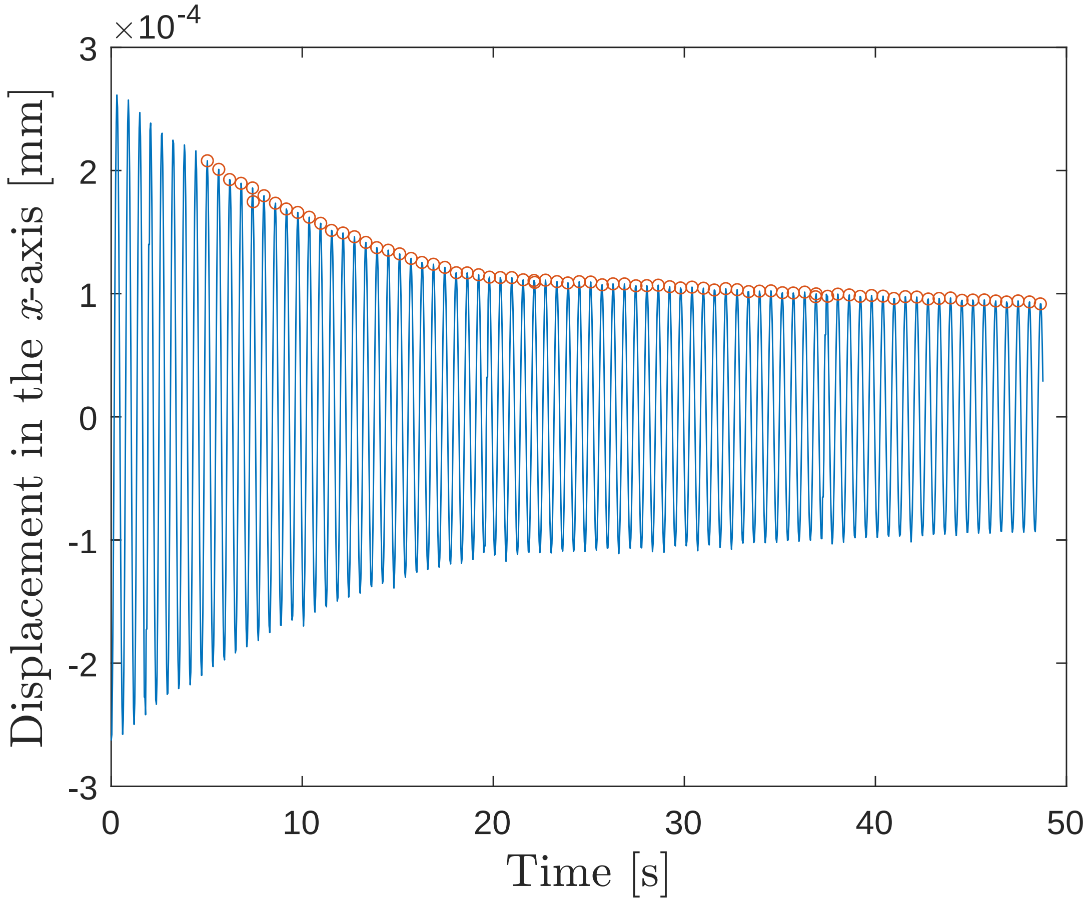


Figure 3.6: Relationship of the displacement in the x-axis and time of the sphere.

3.2 Theoretical results

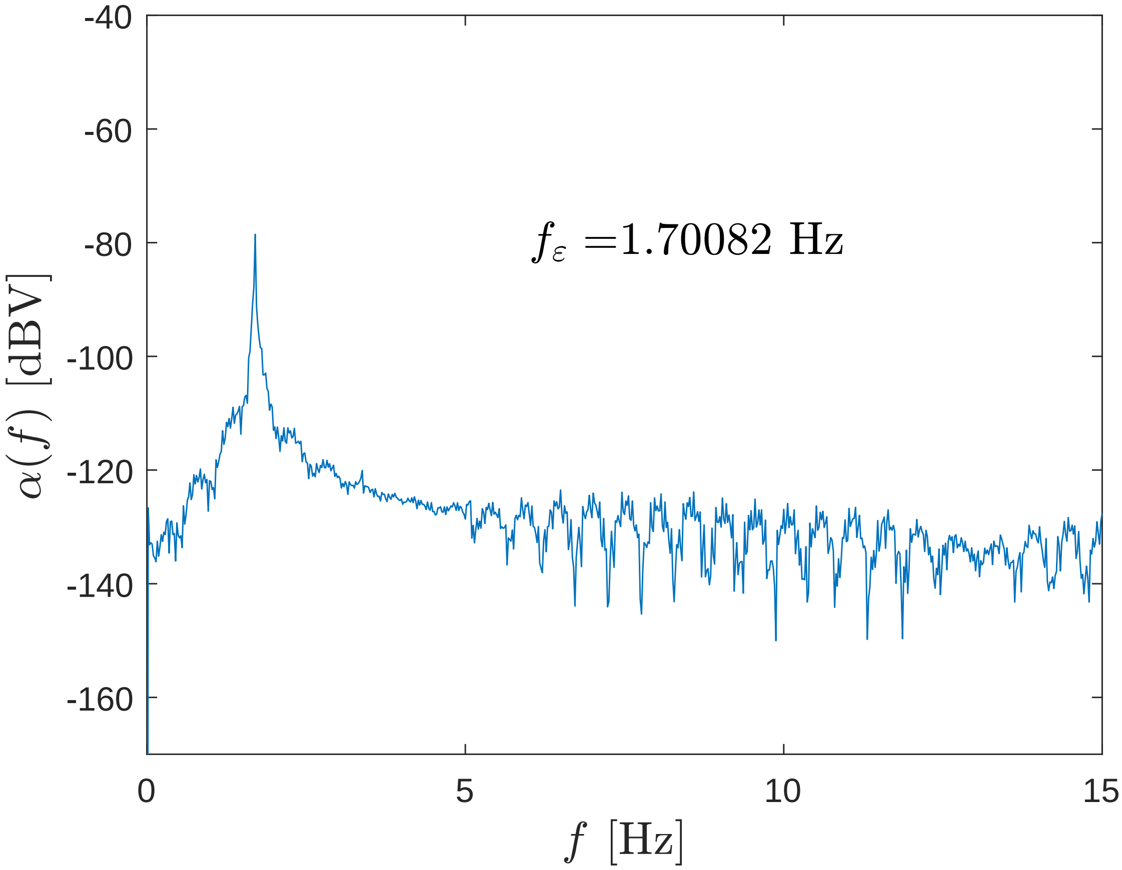


Figure 3.7: Logarithmic scale of the damped frequency of a single sphere with an initial angle of $\frac{\pi}{6}$.

3.2.1 Simple Harmonic Motion

For the analytical results, the definition of a simple pendulum was used as well as its equations.

A simple pendulum, which for this case is constituted by a point of mass hung from a string and is ruled by simple harmonic motion. Deeming that it is a *2D* system,, the mass is attached to the string and it can oscillate (or swing) back and forth in the space when impulsed by, for example, the mass being raised and then, released. If the system is unfettered due the absence of friction or any other dissipation of energy, its motion would continue indefinitely. But in real life, this does not actually happen.

The next analysis and equations can be satisfied for a small swing angle .(Russel, 2011)

The angular frequency is equal to $\omega = \frac{2\pi}{T}$ because every period (duration of time) a revolution or cycle is completed (every 2π). But the natural frequency can also be read as: $\omega = \sqrt{\frac{k}{m}}$.[III]

And since the force the sphere experiences when moving, its displacement obeys Hooke's Law (Sabharishwaran, 2020):

$$F = -ks = -mg\theta = -(\frac{mg}{L})x \tag{3.1}$$

Substituting the restitution constant (Equation 3.1), the natural frequency can be obtained:

$$\omega = \sqrt{\frac{g}{l}} \tag{3.2}$$

Where g is the gravitational acceleration and L is the effective length of the pendulum. [IV] With the dimensions already mentioned above in Subsection 3.1.4 and considering the effective length of $0.102m$, the natural frequency of the pendulum is:

$$\omega_0 = \sqrt{\frac{\frac{9.81m}{s^2}}{0.102m}} \tag{3.3}$$

$$\omega_0 = \frac{9.81rad}{s} \tag{3.4}$$

[III]In Equations 2.3 and 2.4 is explained how this expression is integrated.

[IV]In the Subsection 4.1.1, this expression will be demonstrated in another way.

3.3 Simulations

In this project, as established earlier in the introduction, the principal aim is to expose if the numerical experiments agree with the physical experiments performed earlier. In other words, if the simulations constructed through modeling and programming agrees with what actually happens in real life but also with simulations. And if at the end, all that accords with what is usually taught and understood.

3.3.1 Simulation performed with Matlab

Procedure of the construction of the simulation in Matlab

In this segment, the numerical procedure is going to be described. In *Matlab* a simulation of a resemble the actual case was constructed.

First, the ambient parameters and important features were set; just like gravity, the radio, mass, stiffness coefficient of the spheres, the length of the wires [V], and the time steps.

Then, the spheres were settled by labeling them in the space and specifying that four out the five center of the spheres where at the same height but separated by a distance of $2r = 1.8cm$ in the *x axis*. On the other hand, the last sphere, placed in one of the extremes, had a certain amplitude and therefore a different height. After that, it was established the initial positions and velocities at *zero*, excepting the sphere of the far left ($\frac{\pi}{6}$). To start the motion, it was used the Verlet Algorithm (which will be explained later) in which, at the instant the sphere starts the collision with its subsequent, the velocity and acceleration of the five spheres, changes. With that information, potential and kinematic energy could also be obtained.

The spheres were "labeled" (as seen in the next Figure 3.8) so they could be placed in the space. The sphere located at the furthest on the left side was set with a certain angle at the initial time in order to simulate the releasing of the ball.

[V] The effective length had to be consider as the string length plus the radius of the sphere. In this case, it was of $10.2cm = 0.102m$

The same method was used for the physical experiments.

Additionally, the initial velocities and accelerations were set and a loop was created in order to provide to the other spheres motion, consequently velocity and acceleration as well. The velocities were constructed with the Verlet velocity algorithm.

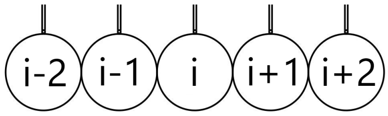


Figure 3.8: Method to label the spheres in the simulations

Verlet Algorithm

This method is a solution for the kinematic equation of motion for any object (Schloss, 2017).This algorithm was introduced in 1967 by the mathematician Loup Verlet. The first version had a lot of accuracy and stability loss. Verlet algorithm does not need explicit velocities.

Advantages:

- Simple
- Modest storage capacity

Disadvantages:

- Moderately accurate

Equations for the Verlet Algorithm

$$x = x_0 + v_0 t + \frac{1}{2} a t^2 + \frac{1}{6} t^3 + ... \tag{3.5}$$

Where x goes for the position, v for velocity, a for acceleration and and error of order $O(\Delta t^3)$.

It also works for the positions:

$$x(t + \Delta t) = x(t) + v(t)\Delta t + \frac{1}{2} a(t)\Delta t^2 + \frac{1}{6}(t)\Delta t^3 + O(\Delta t^4) \tag{3.6}$$

This equation works when it is needed to know the position of the next time step.

$$x(t - \Delta t) = x(t) - v(t)\Delta t + \frac{1}{2} a(t)\Delta t^2 - \frac{1}{6}(t)\Delta t^3 + O(\Delta t^4) \tag{3.7}$$

This equation works when it is needed to know the position of the previous time step.

By solving the equation system with the two previous equations 3.6 and 3.7:

$$x(t + \Delta t) = 2x(t) - x(t - \Delta t) + a(t)\Delta t^2 + O(\Delta t^4) \tag{3.8}$$

This equation means that the position may only be known with the current, the previous position and the acceleration. There is no need of knowing the velocity.

But if possibly at some point it is necessary to compute a term that implicates velocity; just like kinetic energy $(E_k = \frac{1}{2} m v^2)$. The equation right bellow can be applied:

$$v(t) = \frac{x(t + \Delta t) - x(t - \Delta t)}{2\Delta t} + O(\Delta t^2) \tag{3.9}$$

And the equation for the velocity of the next timestep would be:

$$v(t + \Delta t) = \frac{x(t + \Delta t) - x(t)}{\Delta t} + O(\Delta t) \tag{3.10}$$

However, if the velocity is needed to calculate the next time step, the equation bellow cannot be used; so instead, the Velocity Verlet algorithm must be used.

Velocity Verlet

This technique is a slightly corrected version out of Verlet integration and this one provides a better accuracy, stability in the solutions and in some ways, is simpler than the one above. It was purposed in 1985.

$$\begin{aligned} x(t+\Delta t) &= x(t) + v(t) + v(t)\Delta t + \frac{1}{2}a(t)\Delta t^2 \\ a(t+\Delta t) &= f(x(t+\Delta t)) \\ v(t+\Delta t) &= v(t) + \frac{1}{2}(a(t) + a(t+\Delta t))\Delta t \end{aligned} \tag{3.11}$$

$$\begin{aligned} v(t+\frac{1}{2}\Delta t) &= v(t) + \frac{1}{2}a(t)\Delta t \\ x(t+\Delta t) &= x(t) + v(t+\frac{1}{2}\Delta t)\Delta t \\ a(t+\Delta t) &= f(x(t+\Delta t)) \\ v(t+\Delta t) &= v(t+\frac{1}{2}\Delta t) + \frac{1}{2}\Delta t)\Delta t \end{aligned} \tag{3.12}$$

Even this method is much more used, the error has magnitude of $O\Delta t^2$. So, for simulations of many objects that depend on one another (gravity simulation), this algorithm is practical but some crafts must be played in order to scale everything appropriately.

Hence, now the method to describe the velocities is already explained, the next step is to depict how the energy is going to be transmitted from the released ball to the subsequent. As explained above, ideally, the potential energy turned into kinetic of ball *1* should be completely spread to the adjacent sphere but in reality, this is not absolutely true. A certain amount of energy can turn, in this case, into sound or heat. Besides, a total conduction of energy could be viable if it were an Perfect Elastic Collision[VI] and the spheres were treated as a single body or as a collision between two bodies.

[VI] In Subsection 2.9.1, it was explained what a Perfect Elastic Collision is.

The transmission model proposed was also explained. The model $F = -kx^{\frac{3}{2}}$ is more realistic for the interaction of solid elastic sphere since $F = -kx$ provides much less dispersion and predicts relatively simple motions. This model was the Hertzian dynamic force model.

In the code, this force is only transmitted if the distance between two spheres is not different from zero. The force was declared into its x and y components. And so, the position of the spheres, the velocities and the potential and kinetic energy are computed.

3.3.2 Results of of the Simulations performed with Matlab

When the positions in x axis are plotted, it can be clearly identified that when sphere *1* is released and heats sphere *2*, the ball *5* is the one displacing but others also move a bit. And while time elapses, the more the spheres move (also the "static ones").

In the following Figures(3.9, 3.10, 3.11 and 3.12)it was established that:

- The first sphere is blue marked
- The second is purple marked
- The third sphere is yellow marked
- The fourth sphere is orange marked
- The fifth sphere is green marked

Results of the simulations with an initial angle of $\frac{\pi}{6}$

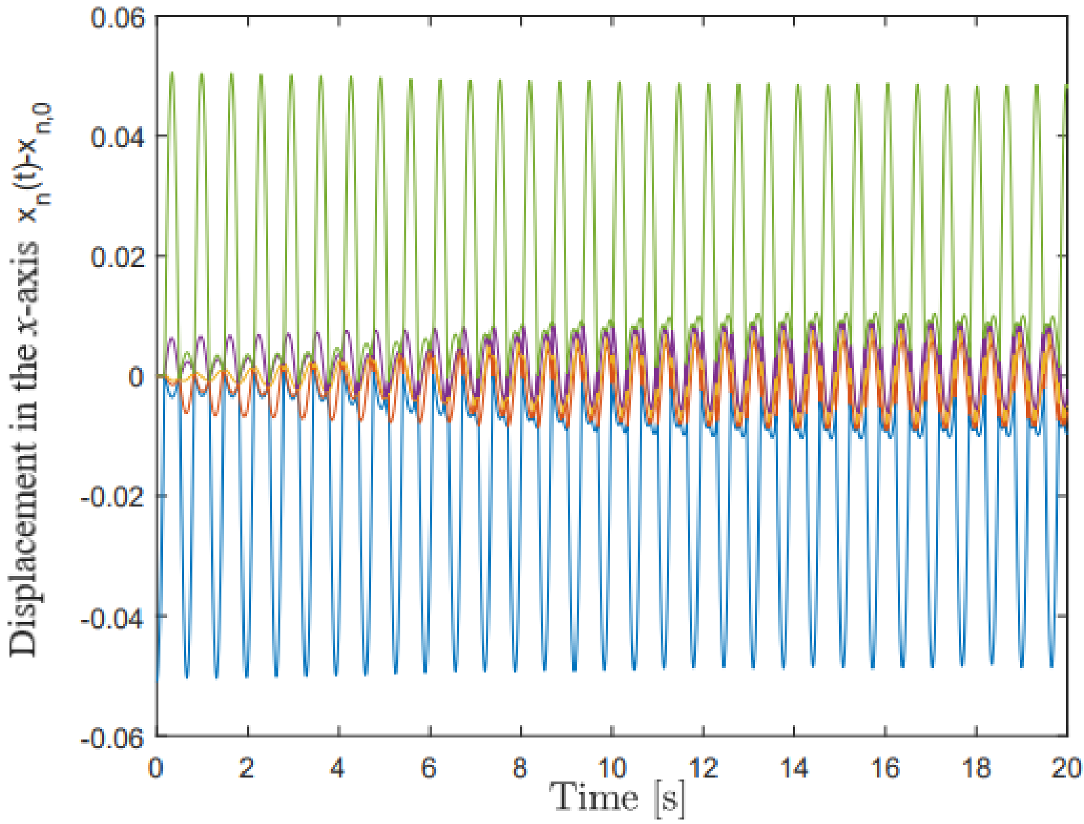


Figure 3.9: Displacements of the spheres in the x-direction, with an initial angle $\theta = \frac{\pi}{6}$, a time step of $\delta t = 0.0001$s and $N = 200000$ steps.

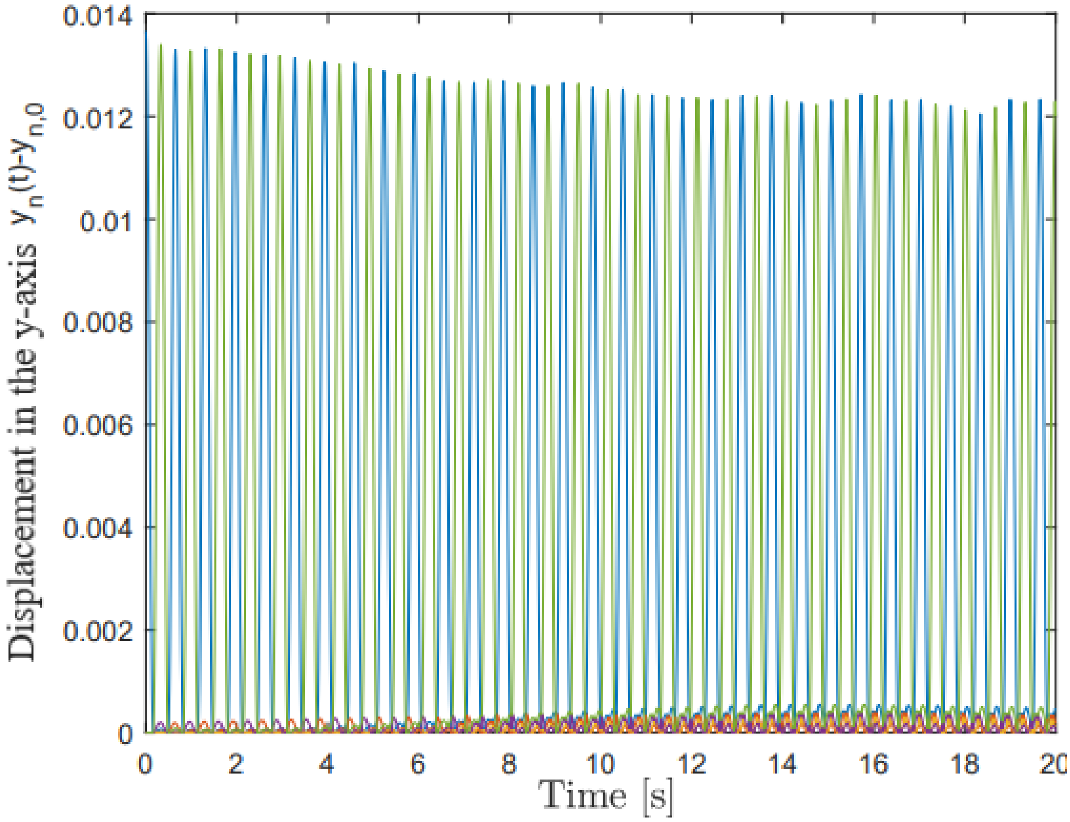


Figure 3.10: Displacements of the spheres in the y-direction, with an initial angle $\theta = \frac{\pi}{6}$, a time step of $\delta t = 0.0001$s and $N = 200000$ steps.

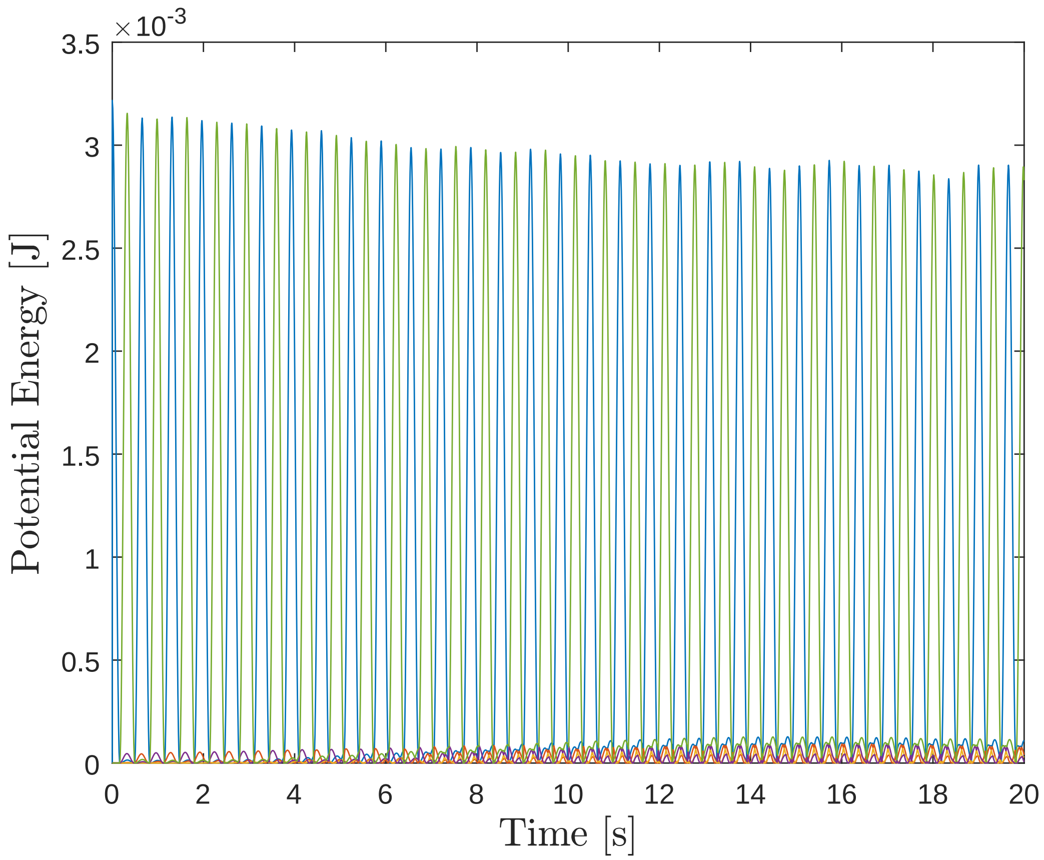


Figure 3.11: Potential energy of the spheres, with an initial angle $\theta = \frac{\pi}{6}$, a time step of $\delta t = 0.0001$ and $N = 20000$ steps.

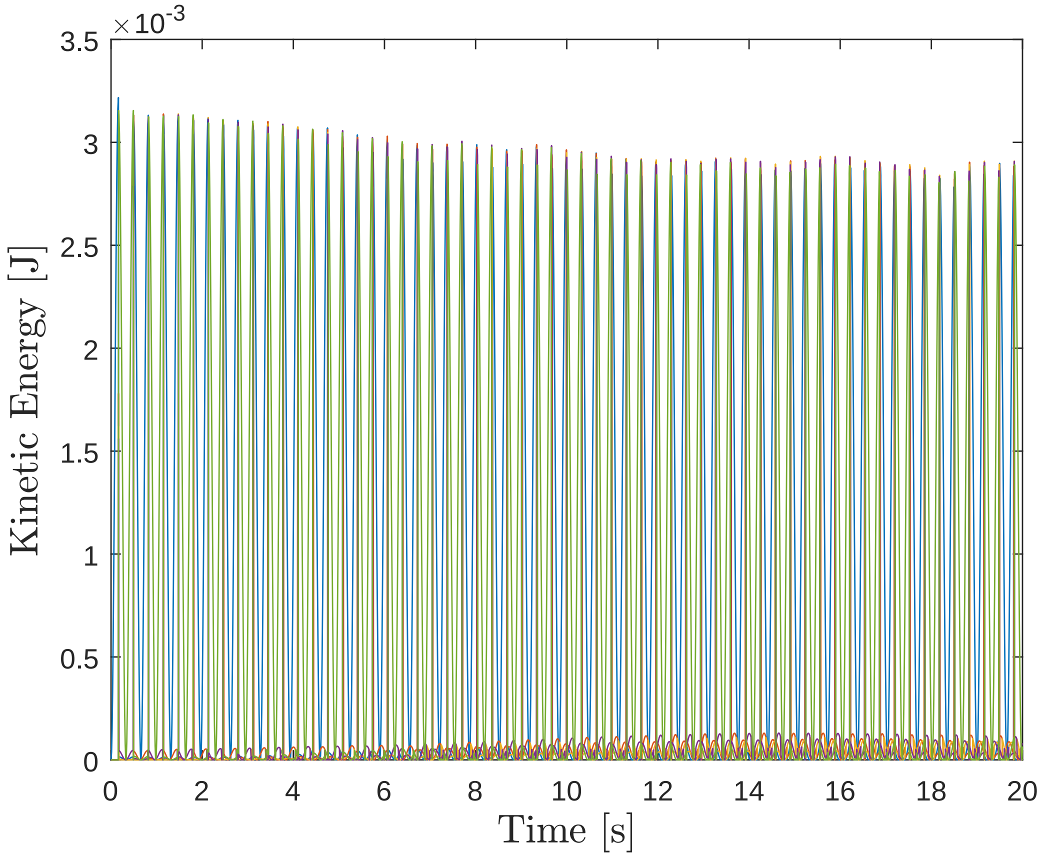


Figure 3.12: Kinetic energy of the spheres, with an initial angle $\theta = \frac{\pi}{6}$, a time step of $\delta t = 0.0001$ and $N = 20000$ steps.

When the potential energy is plotted, it can be seen at the start of the simulation, that sphere *1* has a definite potential energy and sphere *2* and *5* commences to gain potential energy but also, while time elapses, the others gain potential energy too. (See Figure 3.11)

When kinetic energy is plotted, it can be seen that the first sphere gaining kinetic energy is *1* and it is transmitted to sphere *5* and it seems that, at first, the others remain without kinetic energy but, while time elapses, it can be seen that they also gain some energy, not as much as *1* or *5* but they do clearly gain. Because it gives the impression that for the first seconds, the amount of energy for sphere *1* and *5* remains constant between them but as time elapses, the other spheres begin to gain energy, and even the total amount decreases, even the ratio between *1* and *5* remains close to 1 (See Figure 3.12).

3.3.3 Simulation performed with Autodesk Inventor

The other simulation developed was constructed through the software *Autodesk Inventor Professional*. The biggest advantage of it, is the quite interactive, intuitive and flexible way of working with.

Unlike the previous simulation performed with *Matlab*, the whole process with this software was more "visual" since *Inventor* is a modeling software and there was not need of formulate equations for placing the objects in the space, or the equations of motion, or for the contact, etc. At the conclusion section, it will be more deeply explained why this software was chosen.

Procedure of the construction of the simulation in Inventor

Firstly, the device conformed by the frame and the five spheres was built with its corresponding measures, dimensions and material of the actual model used from the physical experiments[VII]. Then, the components were assembled and once in the *Dynamic Simulation* environment, they were given the proper kind of contact. In this case, it was a *3D* contact because in the simulation one important aspect is the transmission of energy through collisions; when a collision needs an area of contact, which can be directly traduced in a *3D* system. The only requirement for the contact between the spheres was to establish between which of the spheres the contact would apply. A joint between the strings and the frame was also set. The frame needed to be fixed so the system could recognize there were only five mobile bodies.

Some important factors in the simulation environment are the forces from the ambient. In this case, the friction between the spheres, the drag force (which can be "understood" as the friction of the ambient air, for this case) and the restitution coefficient. From an article written by *Jane Wang* and *Dong Zhu* (2013), the data for the coefficient of friction were obtained from experiments performed with spherical spheres; just as is the case with this work. One of the results, reported a contact with a Von Mises stress distribution of $v = 0.3$ and a friction coefficient of $f = 0.075$ which was reasonable to conduct the simulation. For the other data, a restitution coefficient of 0.97 was chosen based in the information provided by the article authored by *Viorel Ceanga* and *Yildrim Hurmuzlu* (Ceanga, Hurmuzlu, 2001),who performed a very similar experiment so to complete the simulation, that coefficient was used.

The drag force was implemented taking into account the information presented in *Chapter 2*, specifically from Equation 2.39 contemplating a drag coefficient of *0.5* which is the common value for the air at *"ideal"* conditions .(National

[VII] In the next Chapter, the essential features of the Newton's Cradle at issue are shown.

Aeronautics and Space Administration, 2015)

Once all these elements are adjusted, the next step is to settle the sphere to launch and from which initial angle it will be launched (as can be seen in Figure 3.13). Finally, the software offers the chance of tracking the trajectory and recording the velocity or acceleration of any of the balls. For this project, the tracking of the displacements and velocities of every sphere was recorded and due it, it was possible to prove the decrement in the displacement and the behavior of the velocities of each one of the balls, as it can be observed in Figure 3.14, 3.15, 3.16, 3.17 and 3.18.

3.3.4 Results of the Cradle performed with Inventor

For this simulation, the trajectory and velocities of the five spheres were traced but with an initial amplitude of $\frac{pi}{6}$ and the software was arranged to execute a *3D* contact between the spheres. One simulation were performed as an undamped cradle. The second one was performed as a damped cradle; with air resistance (Drag Force).

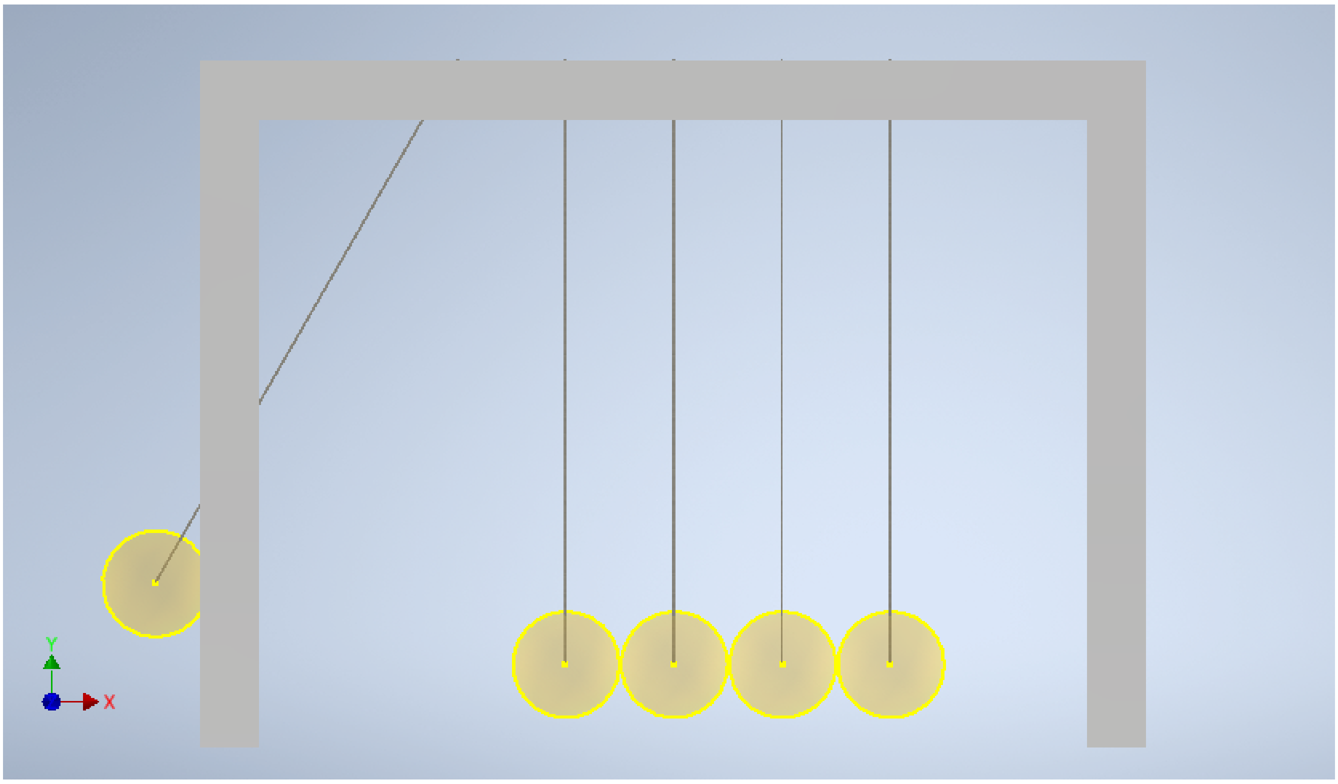


Figure 3.13: Simulation constructed in Autodesk Inventor

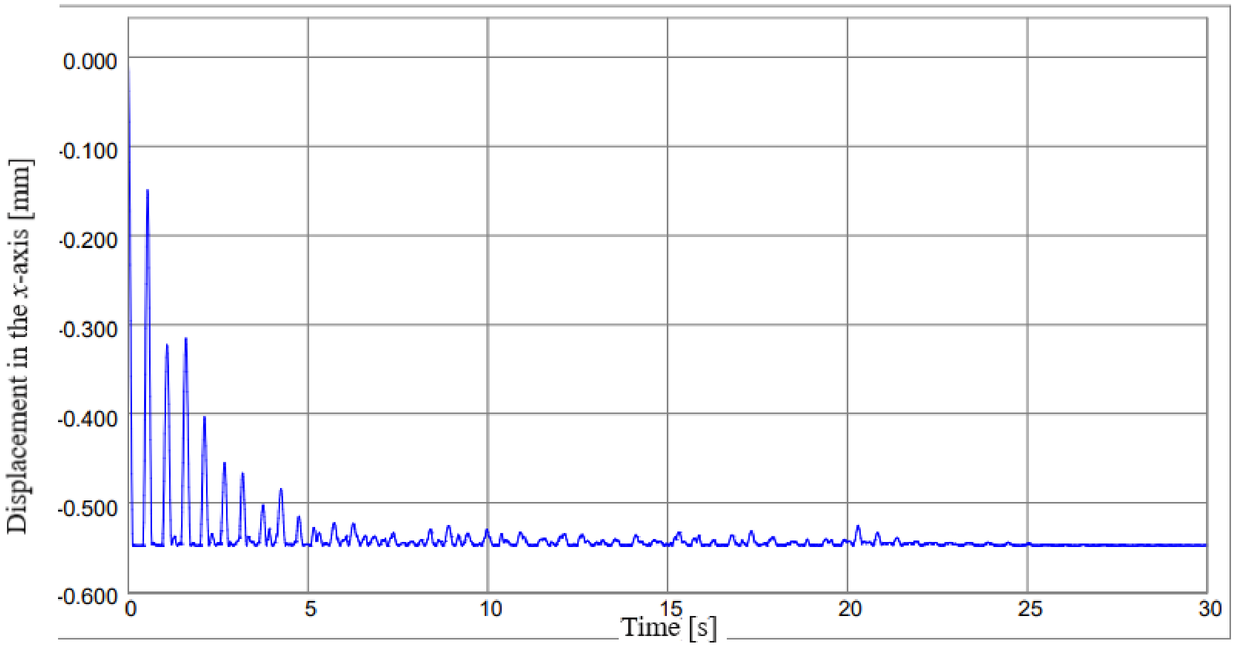


Figure 3.14: Displacement of the released sphere with damping in the x-direction, with an initial angle $\theta = \frac{\pi}{6}$ for $30sec$.

Results of the undamped system with an initial angle of $\frac{\pi}{6}$

Displacements

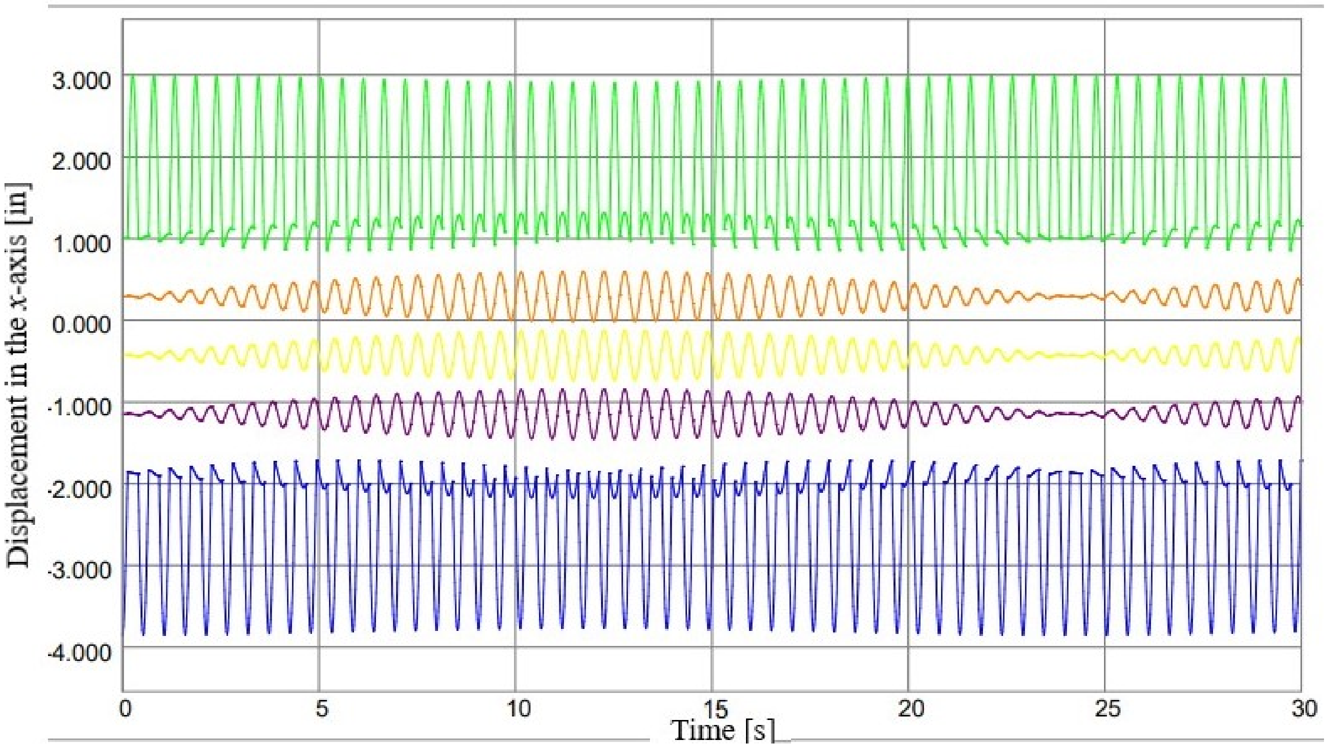


Figure 3.15: Displacements of the spheres without damping in the x-direction, with an initial angle $\theta = \frac{\pi}{6}$ for $30sec$.

Velocities

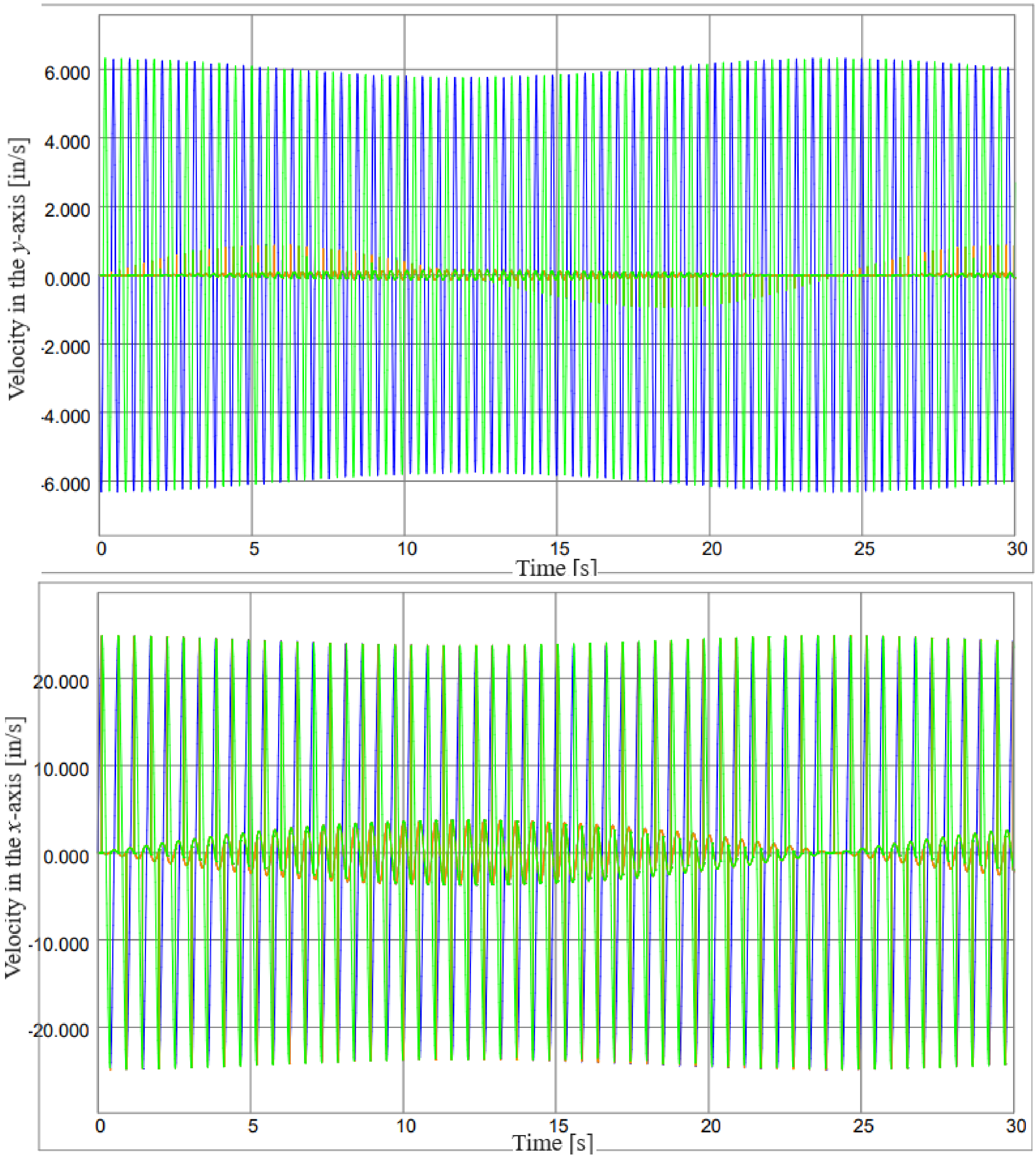


Figure 3.16: Velocities of the spheres without damping in the $x-$ and $y-$ direction, with an initial angle $\theta = \frac{\pi}{6}$ for $30sec$.

Results of the damped system with an initial angle of $\frac{\pi}{6}$

Displacements

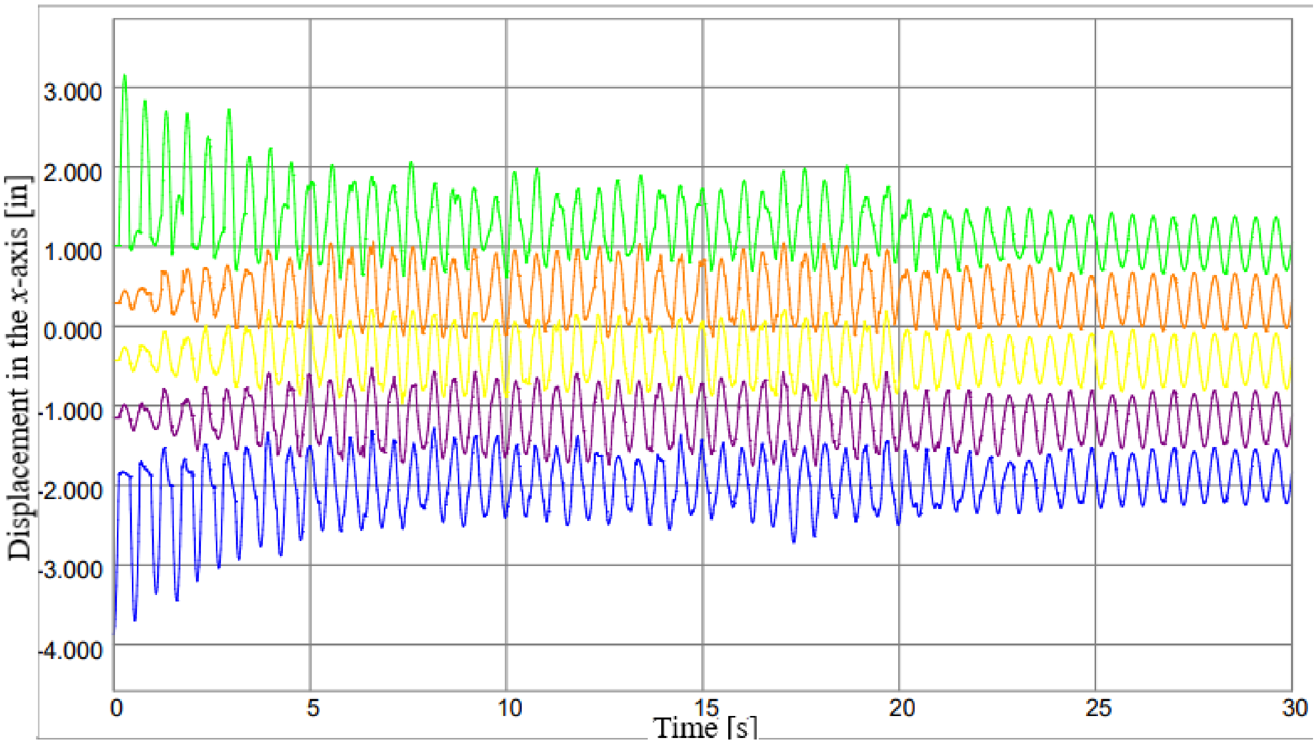


Figure 3.17: Displacements of the spheres with damping in the x-direction, with an initial angle $\theta = \frac{\pi}{6}$ for 30seconds.

Velocities

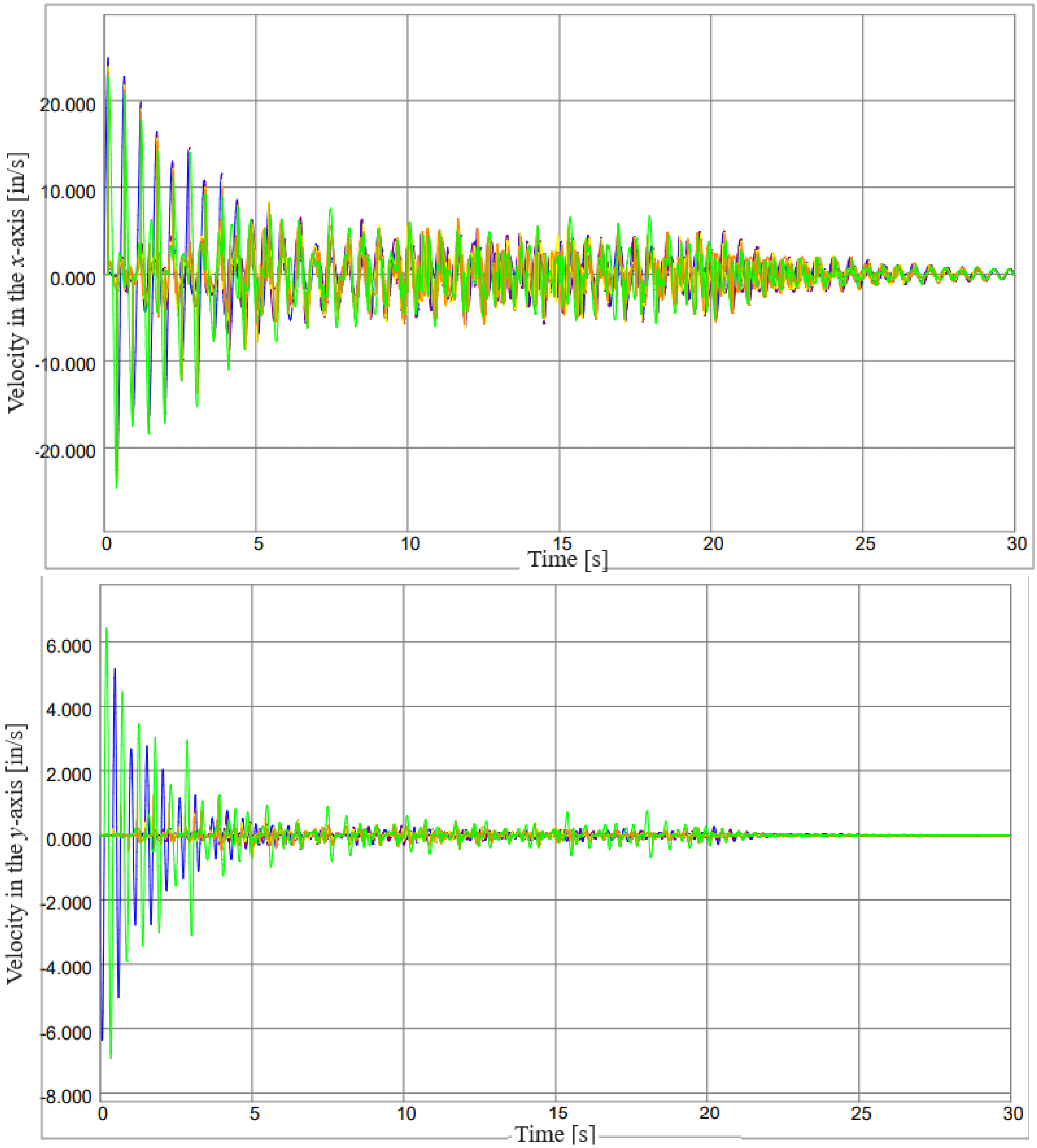


Figure 3.18: Velocities of the spheres with damping in the $x-$ and $y-$direction, with an initial angle $\theta = \frac{\pi}{6}$ for 30seconds.

Chapter 4

Analysis and Discussion

In this section, it will be displayed the examination and the interpretation of the information obtained through both physical experiments and simulations. Thus, in this part, it will be pointed if the results are correct as well as congruent. In other words, demonstrate if the statement determined for this project, is truthful and rightly ascertainable. Firstly, the analysis of part of the theory needed to comprehend the rules present in the collision of five spheres acting as a pendulum will be presented; as well as the results collected through the theoretical information. Then, the outcomes collected from the physical experiments through *Image Processing* will be disclosed and analyzed. And later, the information from the distinct simulations will be discussed.

This analysis was completed as a free vibration analysis since the system worked only with a single impulse.

4.1 Theoretical Analysis of the Results

4.1.1 Obtainment of the Damping Coefficient and Demonstration of its Natural Frequency $\omega = \sqrt{\frac{g}{L}}$

At analyzing a pendulum that is only part of an ideal environment, the main forces present in the system that must be taken into consideration, are gravity and the tension of the string. Now, it will be explained why the angular frequency is equivalent to the square root of the relationship between the gravity and the length of the string and it will be displayed why this definition is important to determine the damping coefficient. The tension will, invariably point back to the rotation point and thus, no torque will be produced. Recalling that to produce torque, it is needed a minimal distance.

On the other hand, gravity generates a restoring torque, so in the equation of the Newton's Second Law of rotation, this term must be inserted. Restoring torque is the kind of torque that causes an twisted or rotating object to return to its original orientation.

$$\sum \tau = I\alpha = -mgL\sin\theta \tag{4.1}$$

The negative sign is due to the torque being opposite to the angular displacement. The rest of the expression has been already explained in Section 2.5.

However, considering that for small angles $\sin\theta = \theta$, the expression can be defined as $-mgL\theta = I\alpha$.

$$\alpha = -\frac{mgL}{I}\theta \tag{4.2}$$

Since the definition of the equation of the Simple Harmonic Motion is: $\alpha = -\omega^2\theta$, the angular frequency can be obtained through this definition.

$$\begin{aligned} &\text{Solving for } \omega \\ \omega^2 &= \frac{mgL}{I} \\ \omega &= \sqrt{\frac{mgL}{I}} \end{aligned} \tag{4.3}$$

I goes for the mass moment of inertia; in this case, of a circle recalling that $I = mL^2$. And therefore (Boston University, no date):

$$\begin{aligned} \omega &= \sqrt{\frac{mgL}{mL^2}} \\ \omega &= \sqrt{\frac{g}{L}} \end{aligned} \tag{4.4}$$

4.2 Analysis of the Physical Experiments

Starting from the theoretical analysis (from Equations 3.1 to 4.4), the evaluation of the physical experiments, can be determined:

Knowing that the critical damping coefficient is defined by $c = 2\sqrt{gL}$. And equating it with the second term of Equation 2.82, the expression for the damping ratio can be obtained:

$$\begin{aligned} 2\omega_n \epsilon &= \frac{c}{m} \\ 2\epsilon\sqrt{\frac{g}{L}} &= \frac{c}{L} \\ \epsilon &= \frac{c}{2L}\sqrt{\frac{L}{g}} \end{aligned} \tag{4.5}$$

$$\begin{aligned} \epsilon &= \frac{c}{2}\sqrt{\frac{L}{L^2 g}} \\ \epsilon &= \frac{c}{2\sqrt{Lg}} \end{aligned} \tag{4.6}$$

According to Equations 4.5, 4.6 and the numerical results, the damping coefficient can be obtained:

$$\begin{aligned} \epsilon &= \frac{c}{c_c} \\ c &= 2\sqrt{gL}\epsilon \end{aligned} \tag{4.7}$$

$$\begin{aligned} c = 2\sqrt{9.81\frac{m}{s^2} \cdot 0.102m} \cdot 0.00169 \\ c = 0.00338 \end{aligned} \tag{4.8}$$

4.2.1 Free Underdamped Motion

The equation that describes the free underdamped motion is show right bellow. But it is only valid when the damping ratio exists between: $0 < \epsilon < 1$.

$$\ddot{\theta} + 2\omega_0\epsilon\dot{\theta} + \omega_0^2\theta = 0$$
$$\text{Its solution can be written as:} \tag{4.9}$$
$$\omega(t) = Ae^{-\omega_0\epsilon t}\cos(\omega_\epsilon t - \psi)$$

Where ω_ϵ is the damped natural frequency which is related to a certain period (T) and the fundamental frequency (f_ϵ).

$$T = \frac{2\pi}{\omega_{epsilon}} \tag{4.10}$$

$$f_\epsilon = \frac{1}{T} = \frac{\omega_\epsilon}{2\pi} \tag{4.11}$$

In order to get the fundamental frequency, with the data of the experiments, the frequency spectrum of the Fourier Transformed was obtained. It was of $1.70082Hz$ as shown in Figure 3.7. With that information, the damped natural frequency could also be computed.

$$\omega_\epsilon = f_\epsilon 2\pi \tag{4.12}$$

$$\omega_\epsilon = 10.6866Hz \tag{4.13}$$

4.2.2 Obtainment of the Damping Ratio

From the logarithmic decrement (Vandiver, no date) the damping ratio equation can be obtained since:

$$\ln\frac{x(t)}{x(t+T)} = \frac{2\pi\epsilon}{\sqrt{1-\epsilon^2}}$$

$$\text{But since } T = \frac{2\pi}{\omega_\epsilon} \tag{4.14}$$

$$\ln\frac{x(t)}{x(t+\frac{2\pi}{\omega_\epsilon})} = \frac{2\pi\epsilon}{\sqrt{1-\epsilon^2}}$$

And solving for ϵ,

$$\epsilon = 1 - \sqrt{(\frac{\omega_\epsilon}{\omega_0})^2}$$

But considering a small damping

$$\sqrt{1-\epsilon^2} = 1$$

$$1 \approx (\frac{\omega_\epsilon}{\omega_0})^2 \approx (\frac{10.6866}{9.9})^2$$

$$1 \approx 1.03 \tag{4.15}$$

Comparing now the physical results with the theoretical ones, they are reasonably exact. The theoretical amplitude only takes into consideration the gravity and the length of the pendulum, not deeming the mass, or the environment, or the amplitude at which the first sphere gets in motion. One should remember that that is because the computation was completed for a simple harmonic motion. In which, precisely nor the angle (because it was taken into account a very small angle), mass or other features are considered.

4.3 Analysis Simulations

4.3.1 Analysis of the Newton's Cradle Simulations performed with Matlab

For this simulation it was established that the the system was going to be free from any kind of external forces, such as air drag, or any kind of impulse; excepting gravity. One of the reasons, it is because, for example, to apply the drag force (in this case, air resistance), an area is needed. And precisely this simulation was created with nothing more than a *2D* type contact. In simple words, it can be said that this simulation would fulfill the function of performing the ideal situation for a Newton's Cradle; this situation is also the most common when teaching and understanding the introduction of this topic. That is why this theory could possibly be the adequate for someone not acquainted to the topic (like a student).

For this project, it was analyzed the case of an initial amplitude of the first sphere with an angle of $\frac{\pi}{6}$. In Figure 3.9, it is shown the displacement of each one of the spheres on the *x-axis* had during twenty seconds.

Despite the simulation did not consider the drag force or another type force, it can be clearly seen that the energy transfer is not the "typical" archetype for an elastic collision. The biggest displacements correspond to the first and fifth sphere (with blue and green color, respectively).

However, it is notable that from the very beginning, the other spheres have a part of that contact energy. And as time passes, there is an increment in the displacement on each sphere (reworded as an increment of energy). But it is also visible, that the difference between the displacement of the spheres placed at the extremes with respect to the other ones, is pretty considerable. For instance, at the beginning (approximately for about 1 and 2.5 seconds), the difference between the first and the second (purple marked) spheres, the *sphere 1* is approximately ten times larger. Whereas, the third sphere (yellow) was the most "stable" since is the one placed exactly at the middle and that left it to have no space to change its position.

At Figure 3.10, at the displacement on the *y-axis* is quite clear that the only ones with "considerable" displacements are the first and the fifth spheres. The displacement of the rest, can be consider as negligible respect to the others. This is because there are only two spheres with a vertical motion, the other's displacement is limited to the *x-axis*.

In the Potential Energy graph (Figure 3.11), it can be noticed that the first and fifth sphere are the ones with the highest potential energy. It is important to remind that this energy depends on the mass, the gravity and a certain distance over a reference plane previously established ($E_p = mgh$). At the simulations and at the physical performance of the experiments, it is evident that, the early mentioned spheres are the only ones with a vertical displacement. So it is normal the great contrast between the pair of the extremes of the cradle and the three remaining spheres.

Recalling that the kinetic energy is the energy that proceeds from motion because is related with the mass of the objects and their velocities; in Figure 3.12 the graph shows that at first, the transmission of energy is larger from the first and fifth but it is not until the last half of time (approximately, 10 seconds) when it is more obvious that *sphere2*, *sphere3* and *sphere4* also acquire some energy. When it was said that in the energy transmission every ball gains a bit, this does not mean that in the relationship between the kinetic energy and time, the amount of the second, third and fourth sphere is considerable. This is because, even it exists a contact, there is no gap and so the their velocity is quite small. Remember that the kinetic energy is the relationship of an object between its mass and its velocity.

As a conclusion of the analysis of the Newton's Cradle generated in *Matlab*, it can be said that the even the five spheres have no gap between them. All of the balls have its respective displacement, but some of them move much more, meaning that the energy transference is distributed between every sphere. Nevertheless, it would be also valid to say that according to the data of the relationship between the time and kinetic and potential energy, the system could represent a complete elastic collision system. This could be valid if the angle of release is very small.

4.3.2 Analysis of the Simulations performed with Inventor

In this simulation, the objective was to demonstrate the statement of this project through the theory of contact, application and considering the system to be damped.

That is why, for this one, it was necessary to carry out two different experiments both with a *3D* contact along the presence of gravity; but one of them with the presence of Drag Force.

It is important to mention that, the type of contact in the system used in this test, was a *3D* contact, however, to avoid redundancies in the system, it was established a gap between each one of the spheres. The slot was pretty small; still, big enough so the software could understand the five spheres were five different, independent and separated bodies. The gap was approximately about $0.254mm$. The gap was added because without it, the software understood that once the released sphere set contact with the second ball, the other three spheres moved along as if they were a single body; while only the first one could move independently. Although, it can be said that, the gap will make the system to behave like a two-body elastic collision scheme. The "error" of the gap can be looked back as the theory reported in the article of Ceanga and Hurmuzlu. (Ceanga, 2001)

Analysis of the Simulations without Drag Force (Undamped System)

To fully explain the analysis, the results from the simulations of *Inventor* will be compared with the experimental results and the cradle constructed in *Matlab*.

In the case of the displacements of the physical experiments and this simulation, there are some divergences although the basis of the phenomenon is similar (both have a decrement at the displacement), however, the reasons are not the same. At the simulation with *Inventor*, it is because there are five bodies setting contact and not just a single sphere. That is where the "*disorder*" comes from. Plus, the conditions of the environment are not the same. The results processed by *Image Processing* had the influence of the air resistance and also, there was no gap between the spheres.

Additionally, this simulation compared with the cradle performed with *Matlab*, one can see that they are practically the same. The way the spheres displaced, are very alike. That is because there is no other external force but gravity in both situations.

Analysis of the Simulation with Drag Force (Damped System)

As shown in Figure 3.17, the displacement of the first and fifth sphere are alike with the simulation of a single sphere performed in *Matlab* (Figure 3.6). But indeed, there are meaningful differences. It is evident, that while in the single sphere test, the displacement is essentially the absolute representation of a simple harmonic motion, with this one, the change of positions at the *x-axis* is shorter with respect to time and it seems to look kind of messy and certainly, it does not look like a simple harmonic motion (because it is messier and much more different compared when the same system has no Drag Force). Even these two situations have mostly the same environment conditions, there is a main difference, because while in one hand there is only a single sphere and in the other, there are five, and a reasoning could be that a simulation of a cradle would be nothing that the multiplication of the behavior of a single pendulum. But, actually, they are five independent bodies.

Recalling Equation 2.40, the most important aspects taken into account were the velocity and the area. The rest, are a constant of the environment. The drag coefficient was taken for the experiment was of $C_D = 0.5$ which is the constant value for a laminar flow for a sphere (Figure 2.8) and the density was no other than the air density at $20°C$.

Now, by comparing the two simulations in *Inventor*, they are quite different, verifying the statement of this project. On one hand, there is an ideal system (referencing only about external forces), in the other hand, there is the representation of what would most likely to happen. This is quite clear in Figures 3.15, 3.16, 3.17, and 3.18.

It would be important to mention that, in Figures 3.15 and 3.17, there is the absolute position of the each one of the spheres, that is why they look as if it were separated.

Chapter 5

Conclusions

In very general words, for this project, the aim was to, firstly understand what a pendulum is, how does it work, what parameters are important at its performance, and then, couple all that knowledge and being able to introduce and explain the Newton's Cradle. All this, so that in the end, as consequence, the explanation of a collision between several bodies could be analyzed and understood easier. Indicating, at the same time what it is true and what not so true about the theory of contact and energy transference for example.

Describing part of the applications and the history of the pendulum (in the second chapter), was pretty helpful to realize that a "*simple*" mechanism is very relevant and quite present in everyone's daily life. After that, it was important to show the whole physical framework behind a pendulum, what kind of motion does it have, the equations that are necessary when an small angle is applied, what happens if the environment offers friction and the numerical solution when there is no a small angle to perform with. Additionally, some fluid mechanics concepts and theory of contact (Hertzian contact) concepts were explained to complete and clarify a greater interpretation about a generally difficult topic: Damping.

Then, in the third chapter, it can be found the record of the experiments, both of the physical experiments and the simulations. For both, it was described the material needed to perform the experiment with, a quick description of the exercise.

In case of the physical tests, it was needed to report important features of the device because there were going to be the same parameters used at the simulations. Principal characteristics of the cradle just like the diameter of the spheres, or the length of the string attached to the frame and the sphere, hardness of the spheres, its mass, among other, were fundamental in order to be able to compare and discuss the analysis in the corresponding chapter.

Then, the method of *Image Processing* was also discussed while the needed material were also described. After that, the results were presented. It can be found how the natural angular frequency and the damping ratio were obtained, the displacement the sphere had during the experiment (Recall the test was only performed with one sphere) and the fundamental frequency as well as the Fourier Transform.

It would be important to mention some problems which took place during the performance of the experimentation. Just like, for example; the quality of the camera; causing that at the reading of the video, in some frames, the sphere was not highly recognized so that made the centroids resulted to be altered. Another problem was that at the graph of Displacement-Time, some of the peaks detected were smaller that their neighbor (the successive one), obviously those were errors because that is not possible.

A recommendation at recording would be: If the illumination is not the adequate to spotlight the sphere, a non-darkened object (resembling a reflector) can be placed in a way it will work as a "natural flash". For this case, the objects used were two pieces of styrofoam and a sheet of paper placed aside and under the cradle respectively as seen in Figure 3.2.

As closure about the physical experiments, it can be said that, the results illustrate the behavior of a simple harmonic motion. This argument is ascertainable in Figure 3.6 by looking the displacement the sphere had through time.

As conclusion about the analysis between the results obtained from physical experiments and the theoretical analysis, it can be recognized that the numerical results with respect to the simulations, had a great accuracy. The results agreed with the expected ones. Even some troubles were present, the results could be obtained. The differences between the two methods relapse in the equations and procedures. With that, it is demonstrated that even for different procedures, the outcomes are equal or at least, very similar.

But would also be valid to say that the discrepancy between the results may be due, besides the previously mentioned, to an error of human factor. For example, for the *Image Processing*, the code asked for the diameter of the sphere and once with the information, the first frame of the video appeared and it was necessary to select from where to where the sphere was located, so *Matlab* could take those parameters and binarize every single frame. In other words, every time at the code, it was stated that a contour of a circle had to be found, only that area was "*painted*" of black (the code collected this as "1") and the rest of the frame was taken as white (the code collected this as "0").

So, if right before start recording, the hand handling the sphere moved, the first frame of the video could be affected to such a point, the sphere had a different initial amplitude; or maybe it is possible that a part of the hand is visible and so that the system would understand that the there is not only a circle to capture but also a unknown geometrical shape.

Another error, would be if at some point of the recording the site were the cradle is placed or the camera is moved. These three possible scenes may lead to mistakes during the process.

Now, about the results obtained in the Cradle simulation performed *Matlab*, it can be said that with the assumption of $F = -kx^{\frac{3}{2}}$ that was used, the system lead to non zero velocities for every single sphere just after the first collision.

Specifically in the Matlab Cradle's simulation, there were set the gravity, radio of the spheres, the length of the wires and the time differential (δt) and the number of steps (N). The simulation lasted twenty seconds with a δt=0.0001 and $N = 200000$ steps.

The differential was adjusted with a trial-error method, with a bigger differential, the system would turn uncontrolled. Once the spheres were settled, their initial positions and velocities were set at zero. Except, the sphere to be released. To solve the contact between the set of spheres and the velocities, it was used the Verlet algorithm and the Velocity Verlet method. At the corresponding section (3.3.1), it was explained why that method was chosen. To complete that part, the results of the displacement of each ball in the $x - axis$ and the $y - axis$, the potential and kinetic energy along with their interpretations were presented.

Coming after, the simulation carried out with *Autodesk Inventor* was shown. At the beginning, the software was introduced and here are some of the reasons why this program was chosen.

Advantages:

- The sketching and assembling operations and tools needed for the project are uncomplicated
- At the Dynamic Simulation, the constraints of motion and some other important features such as gravity was easy to add and arrange.
- The response time was very short which made possible to obtain the results much faster and in case of a mistake, the correction could be completed very quick.

- Specifically for this project, the data from each one of the spheres was acquired in a very simply way.
- The span of time for the simulation was very reasonable and if a change was necessary to apply, the software allows to do it without any complication

Disadvantages:

- At performing a dynamic or a stress analysis or a force or torque; it is difficult to the software to understand that the force is applied to the whole body, in this case, to a sphere. Searching in internet forums, it was found that this a specific problem of *Inventor*.
- When the simulation was performed without any kind of gap between the spheres, the system understood there were two bodies. The first one (the released ball), as soon as it touched the adjacent sphere, this second ball would move but along the other spheres, so the first body would be the sphere *1* and the second, the spheres *2*, *3*, *4* and *5*.

One interesting fact about have performed simulations in *Inventor*, was that apparently, bouncing backward and residual motion was present during the performance. The bouncing backwards was detected because the software allows that during a simulation, the trajectory of the body can be displayed. and the residual motion was detected when at some moments, the spheres moved "more or less" than they should have. According to (Gavenda,1993),the error can be attributed to experimental defects and that, if there was left a gap (even it was a pretty small gap), it is the reason for the bouncing backward.

Lastly, now that the results of both methods (physical and numerical) has been analyzed, it can be said that, the effect the damping has in this particular system although is relevant, is not the most important; as seen in the results, because it is very small, and because of that, the system turns into a **small damping system**.

In summary, what can be concluded from this project is that, teaching perfect elastic collisions using as an example, a Newton Cradle, it is wrong or at least it is not the complete outlook, maybe it can be used as an example at the introduction of the topic yet remarking and making clear that the cradle would function in a certain way under an **unreal** background.

In the same way, it is crucial to say that when it comes to collisions of several bodies, it is difficult to express that the behavior can be accepted as tidy and or predictable.

Finally,to end this project, it would be important for the current and future models concerning about contact of bodies, to precise that the momentum, accelerations, velocities and the displacements within the bodies either planted in a Newton's cradle or in any body's collision arrangement, suffer changes, and so, those changes must not be conceived like a prediction, but a fact.

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