

## 4. SHAFTS

A shaft is an element used to transmit power and torque, and it can support reverse bending (fatigue). Most shafts have circular cross sections, either solid or tubular. The difference between a shaft and an axle is that the shaft rotates to transmit power, and that it is subjected to fatigue. An axle is just like a round cantilever beam, so it is not subjected to fatigue.

Shafts have different means to transmit power and torque. For example, it can use gears, sprockets, pulleys, etc., and also have some grooves to keep these elements rigid and avoid their vibration, such as key seats, retaining ring grooves, etc. Also, to be able to avoid vibration of the elements, and assure an efficient transmission of power and torque, some changes in the cross-section of the shaft can be made.

All these elements, because of their weight, of the torque they are transmitting, and of the grooves made to avoid excessive axial movement, produce stresses, and as a consequence, strains. It has been demonstrated that stresses and strains are directly related if they are in the elastic range (by Hooke's law), and that if a strain is produced, then as a consequence a stress has to be produced, and vice versa. Also, there are four types of shafts, and each one of them has a different way to calculate their corresponding stresses. In this chapter, there will be analyzed four cases: uniform shaft, multiple-section shaft, variable-cross-section shaft and relative rotation. All of these shafts are subjected to torsion, so we will review first the concepts of it.

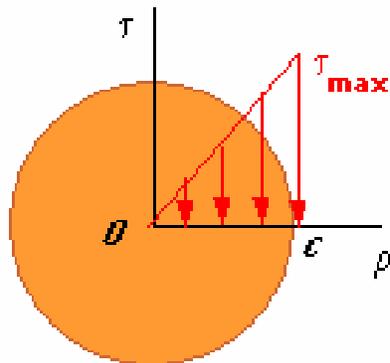
“Torsion refers to the twisting of a straight bar when it is loaded by moments (or torques) that tends to produce rotation about the longitudinal axis of the bar.” (Gere & Timoshenko, 1997). As demonstrated in the Chapter 3, torsion produces an angle of twist in the shaft, and this angle of twist produces a strain, which varies linearly with the distance from the center of the shaft to the surface. As a consequence of the strain produced by the torsion, also some stresses will be produced, which are shearing stresses. If we are working in the elastic range, then these shear stresses can be determined from Hooke’s law in shear:

$$\tau = G \cdot \gamma$$

where  $\gamma$  is the shear strain, and  $G$  is the modulus of rigidity (or shear modulus of elasticity). If we combine this equation with the equation to calculate the shear strain we will get that

$$\tau_{\max} = G \cdot r \cdot \theta = G \cdot \rho \cdot \theta = \frac{\rho}{r} \tau_{\max} \quad ^1$$

where  $\tau_{\max}$  is the maximum shearing stress, which occurs at the surface of the shaft (where  $\rho = \rho_{\max}$ ) as seen in Figure 14.



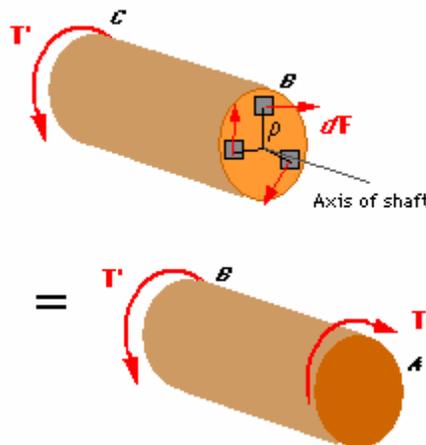
<sup>1</sup> From Gere & Timoshenko, *Mechanics of Materials*, PWS Publishing Company, United States, Fourth Edition, 1997, p. 192

Figure 14 15 (Beer & Johnston et. al. Interactive Tutorial, 2001)

What is seen in Figure 14 is how stresses act on the plane of the cross-section, but this is not the only plane where they act. They are accompanied by shearing stresses of the same magnitude, but these ones act on a longitudinal plane. This comes from the fact that “...equal shear stresses always exist on mutually perpendicular planes...” (Gere & Timoshenko, 19797).

Now, we have to take a look to the torsion formula, which states the relationship between the torque and the shearing stresses. These stresses are acting continuously on the cross-section of the shaft, and because of this they produce a resultant in the form of a moment. This moment is equal to the torque applied to the shaft. Considering Figure 15 (which represents a cut in the middle of a bar with an applied torque), in can be seen that shearing forces are produced, and that they can be calculated at any distance  $\rho$  from the axis. If we make the sum of the moments around the axis, then we will get

$$T = \int \rho \cdot dF ,$$



and as the force is related to the shearing stress by  $dF = \tau dA$ , then the torque will become  $T = \int \rho(\tau dA)$ <sup>2</sup>. Finally, the shearing stress can be obtained from  $\tau = \frac{T \cdot \rho}{I_p}$ , where

$T$  is the applied torque,  $\rho$  is the distance from the axis to the point where the shearing stress is being calculated, and  $I_p$  is the polar moment of inertia ( $I_p = \frac{\pi \cdot d^4}{32}$ ). If the shaft

is hollow, then the formula used is the same:  $\tau_p = \frac{T \cdot \rho}{J}$  where  $J$  is the polar second moment of area (which is the same as the polar moment of inertia), but it is calculated

$$\text{as } J = \frac{\pi}{32} (D_0^4 - D_i^4).$$

An important thing for the designer is to identify the direction of the shearing stresses in the shafts produced by the applied torque. Now it is known that to calculate the magnitude of the shearing stresses at any point from the axis to the surface, the formula  $\tau_p = \frac{T \cdot \rho}{J}$  can be used. To establish the direction of them, a cut should be made in the middle of the shaft and it must be perpendicular to the longitudinal axis. If the two parts of the shafts are now “separated”, each one must have the same shearing stresses in that plane, but in opposite direction (see Figure 16). This is because of equilibrium. At the beginning, the direction of the shearing stress is assumed. By taking moments again about the axis of the shaft, the shearing stress must be balancing the applied torque. Once the static equilibrium is established, the direction of the shearing stress is known. The

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<sup>2</sup> From Beer and Johnston et. al., *Interactive Tutorial*, Chapter 3, 2001

stresses acting on the surface of a shaft are as shown in Figure 17. They act following the path of the perimeter of the circle in one direction. If the torque is applied on the opposite direction,

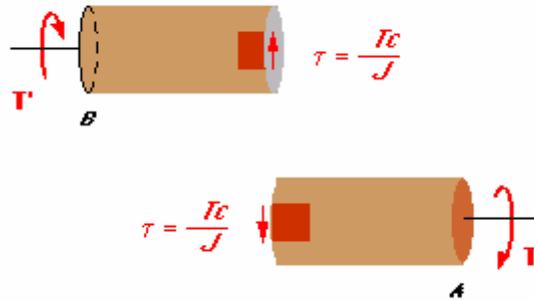


Figure 16 (Beer & Johnston et. al. Interactive Tutorial, 2001)

then these stresses will be acting also in the opposite direction, following the same principle stated before to establish the static equilibrium between torque and stresses. The complete description of how stresses are behaving in some points of the shaft is seen in figure 17. At each point, there are two stress free surfaces, and the other four surfaces have stresses acting as shown.

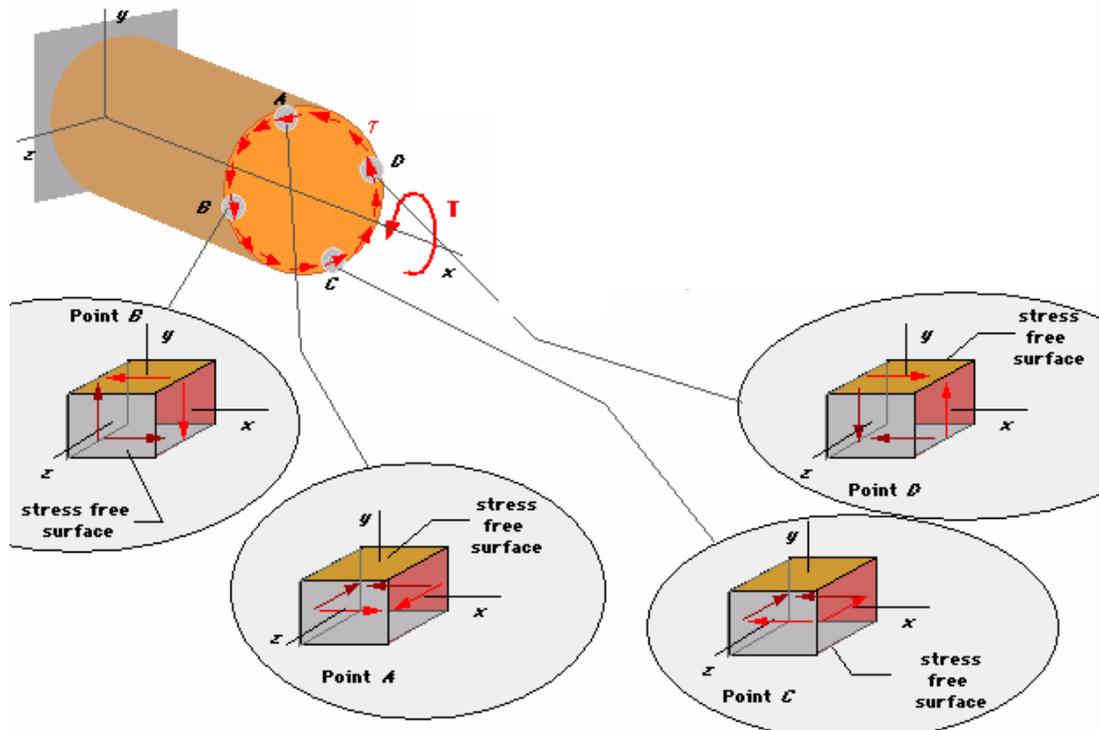
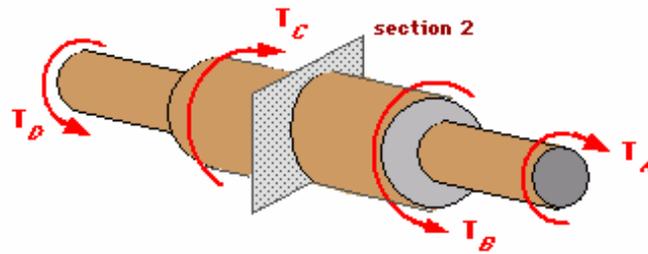
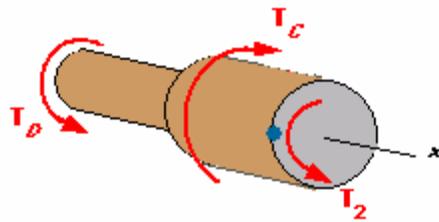


Figure 17 (Beer & Johnston et. al. Interactive Tutorial, 2001)

If we have a stepped circular shaft with some torques applied, then the principle of static equilibrium must be accomplished too, and this way we will be able to find the magnitude and the direction of the inner torque at each section, and with this we will be capable of calculating the shearing stresses in that particular section. For example, in Figure 18a it is seen that we have a shaft with four different torques applied, and that it is cut on one section of the shaft (where the plane of section 2 is) (Figure 18b). By applying the



(a)



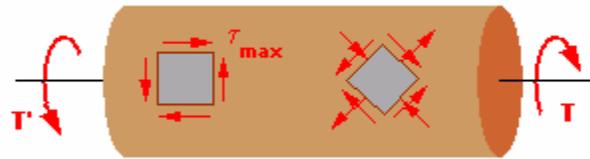
(b)

Figure 18 (Beer & Johnston et. al. Interactive Tutorial, 2001)

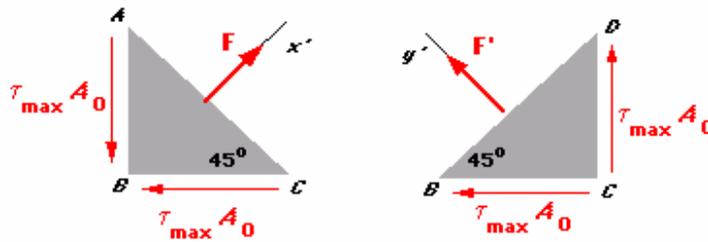
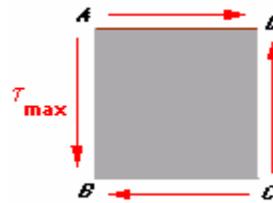
static equilibrium principle, then we would have that the torque 2 ( $T_2$ ) is equal to the summation of the other two torques:  $T_2 = T_D - T_C$ . Once this torque  $T_2$  is known the stresses in that section can be calculated. It is important to understand that the torque  $T_2$  is only valid for that section, and that it describes only the inner torque for that section. If we want to calculate the inner torque of section 1 (the section on the left before the step), then we would have to establish again the static equilibrium equation, and calculate the corresponding torque to calculate the corresponding stresses produced by it.

So far, we have seen that a normal and shearing stresses combination can occur under the same loading conditions, and they will be oriented at some angle from the axis of the shaft. If an element oriented at  $45^\circ$  to the axis of the shaft is considered (Figure

19a), then the normal and shearing stresses will be on inclined planes. To determine the magnitude of the stresses, we must consider two triangular elements (Figure 19b). As we can see, the vectors resulting from forces AB, BC and CD are the maximum shearing stresses ( $\tau_{\max}$ ). The normal stress is the one which is expressed as  $F$ , and as seen it must be



(a)



(b)

Figure 19 (Beer & Johnston et. al. Interactive Tutorial, 2001)

perpendicular to the surface. To determine the magnitude of the forces producing the normal stress, we have to use free body diagrams<sup>3</sup>, from where we would find that

<sup>3</sup> To see the demonstration, take a look at: Beer & Johnston et. al., *Mechanics of Materials*, McGraw-Hill, New York, Third Edition, 2001, p. 142

$$F = \tau_{\max} A_0 \sqrt{2}$$

$$F' = -\tau_{\max} A_0 \sqrt{2}$$

To obtain the magnitude of the stresses, we just have to divide this force over the cross-sectional area of each face. From here we can conclude that the normal stress at  $45^\circ$  is  $\sigma_{45^\circ} = \pm \tau_{\max}$ . If the element is not inclined, then it will be experiencing pure shear, while if it is inclined on some plane respect to the longitudinal axis of the shaft, it will be experiencing normal stresses (tensile on two faces and compressive on the other two faces) and shearing stresses.

Now it is time to analyze the four types of shafts described in the beginning of the chapter. We will start with the uniform shaft. When talking about strains, we have seen that the equation to calculate the maximum shearing strain in a uniform shaft of length  $L$  and radius  $c$  is  $\gamma_{\max} = \frac{c\phi}{L}$ , where  $\phi$  is the angle of twist. Another thing we have seen is

that the equation that relates the torque with the maximum shearing stress is  $\tau_{\max} = \frac{T \cdot c}{J}$ .

If despite of the torque we are applying to the shaft it still continues working in the elastic range (which means that the maximum shearing stress does not exceed the yield stress),

then Hooke's law can be applied, and we would get that  $\gamma_{\max} = \frac{\tau_{\max}}{G}$ . From here,

through a few substitutions<sup>4</sup> with the equations presented before, we can get

that  $\gamma_{\max} = \frac{T \cdot c}{G \cdot J}$ , and if we solve for the angle of twist, we get that

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<sup>4</sup> Beer & Johnston et. al., *Mechanics of Materials*, McGraw-Hill, New York, Third Edition, 2001, p. 150

$$\phi = \frac{T \cdot L}{G \cdot J}$$

Here we can see the relationship between the angle of twist and the torque, the length of the shaft, the modulus of rigidity and the polar moment of inertia. In all of these cases, the angle of twist is proportional to the other factors.

In the case of a multiple-section shaft, we can calculate the angle of twist as stated before by making cut planes in every section of the shaft, and then applying the principle of static equilibrium. In this way, we will be able to determine which the angle of twist of each section is, and the total angle of twist is going to be the summation of all the  $\phi$  of the sections. In each of these sections, the four parameters to calculate the angle of twist are not going to be constant. If the shaft is made of two materials, then it won't have the same modulus of rigidity. If different torques are applied to each section of the shaft, then the torque is not going to be constant either. Also, the length can change, as well as the polar moment of inertia, which varies depending of the radius of the section. So, if we want to express the total angle of twist of a shaft including n number of sections, then it will be expressed as

$$\phi = \sum_{i=1}^n \frac{T_i \cdot L_i}{G_i \cdot J_i}$$

To study how the angle of twist is in a variable cross section shaft, we must first take a look at Figure 20. As the cross section is varying along the axis, we must make an integration of all the small elements contained in the shaft. For this, take a

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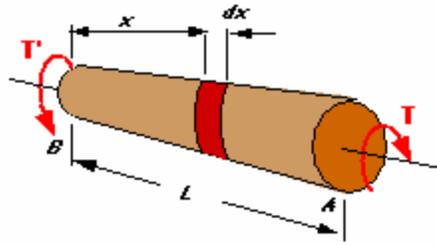


Figure 20 (Beer & Johnston et. al. Interactive Tutorial, 2001)

small disc (which in Figure 20 is taken as  $dx$ ). This small portion of the shaft is going to be taken as the length of a small disc. Once this is done, the equation for the angle of twist would be  $d\phi = \frac{T \cdot dx}{G \cdot J}$ . It is important to understand that the polar moment of inertia is a function of  $x$  (at each different point of the shaft the radius is different), and that this function has to be determined. So, if we want to obtain the total angle of twist of the shaft, we have to make a summation of each small disk (by making an integral). This integral has to be over the length of the shaft (which gives us the limits of the integral). The expression to calculate the total angle of twist is

$$\phi = \int_0^L \frac{T \cdot dx}{G \cdot J}$$

An alternative form of this equation is  $\phi = \int_0^L \frac{T(x) \cdot dx}{G(x) \cdot J(x)}$ , where we can see that each term of the equation is a function of  $x$ .

Finally, we have one of the most common cases of application of shafts: shafts with relative rotation. By this concept, we understand that both ends of the shaft are rotating. “In this case the angle of twist of the shaft is the angle through which one end

rotates with respect to the other.” (Beer & Johnston, 2001). Consider Figure 21. By assuming that each shaft is elastic, and that they have length  $L$ , radius  $c$ , and modulus of rigidity  $G$ , if a torque

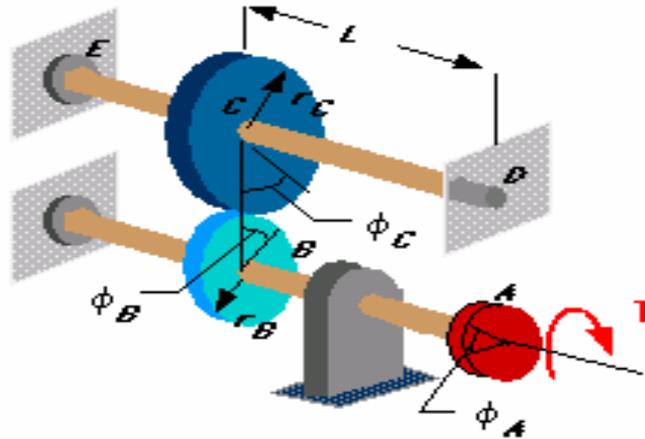


Figure 21 (Beer & Johnston et. al. Interactive Tutorial, 2001)

$T$  is applied at  $A$ , then both shafts will start rotating. As seen in the Figure, the end  $D$  is fixed, so this will cause an angle of twist which will be measured by the rotation  $\phi_c$  of end  $C$ . In the other shaft, both ends are rotating, so the rotation of the shaft is going to be the difference between the angles of rotation  $\phi_b$  and  $\phi_a$ . This is known as relative rotation.

Knowing this, we can get the equation for relative rotation:

$$\phi_{A/B} = \phi_A - \phi_B = \frac{T \cdot L}{G \cdot J}$$

To end this chapter, Figure 22 is going to be presented, where we can see an example of how each kind of shaft is: uniform, multi-section, variable cross-section or relative rotation. Each case has been analyzed, and this Figure is only to understand which the composition of each case is: which ends are fixed, which ends are rotating,

how is the cross-section of the shaft along the longitudinal axis, or if it is transmitting torque from one shaft to another with power transmission elements like gears, spurs, pulleys, etc.

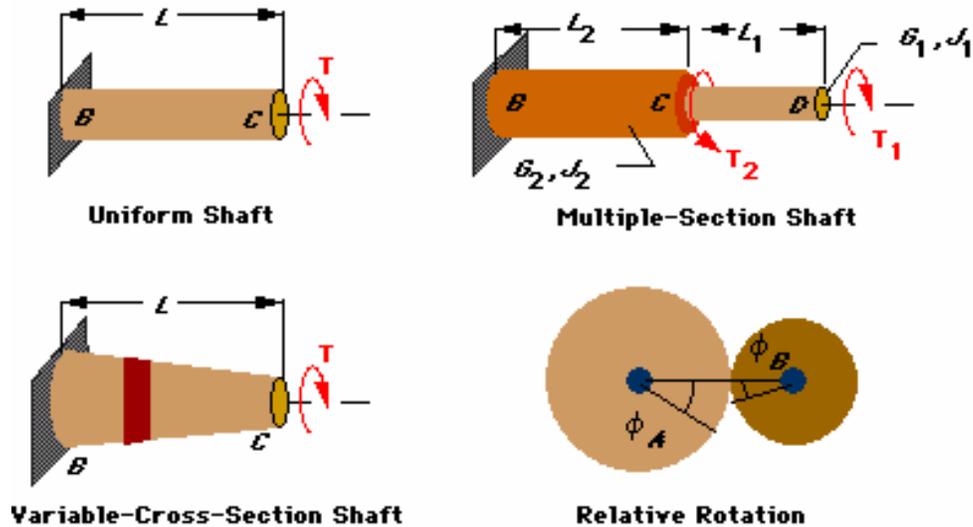


Figure 22 (Beer & Johnston et. al. Interactive Tutorial, 2001)

- **Safety Factor**

An important thing to consider when designing any engineering element is the safety factor. “The maximum load that a structural member or a machine component will be allowed to carry under normal conditions of utilization is considerably smaller than the ultimate load.” (Beer & Johnston, 2001). This smaller load is known as allowable load, working load or design load. So, when an element is working (transmitting torque, for example), it is going to be working safely until a limit (the allowable load), and only a portion of the ultimate-load capacity is going to be used. The remaining part of the capacity of the element to support load is “kept in reserve” to assure that the element is

working safely. “The ratio of the ultimate load to the allowable load is used to define the factor of safety.” So the equation to calculate it is

$$\text{Factor of safety} = F.S. = \frac{\text{ultimate load}}{\text{allowable load}}$$

To choose the correct factor of safety, some considerations are required<sup>5</sup>:

1. Variations that may occur in the properties of the member under consideration.
2. The number of loadings that may be expected during the life of the structure or machine.
3. The type of loadings that are planned for in the design or that may occur in the future.
4. The type of failure that may occur.
5. Uncertainty due to methods of analysis.
6. Deterioration that may occur in the future because of poor maintenance or because of unpreventable natural causes.
7. The importance of a given member to the integrity of the whole structure.

The factor of safety for shafts is 3. This factor was chosen by considering the variations in material properties, the effects of size of the shafts, the type of loading, the effect of machining or forming, the effect of heat treatment, the effect of shear on function, the effect of operating environment, the specific operating requirements and the concern for the human safety.

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<sup>5</sup> Taken from Beer & Johnston et. al., *Mechanics of Materials*, McGraw-Hill, New York, Third Edition, 2001, p. 29