3. STRAIN

In the past chapter, we have talked about the concept of stress, but there is also another concept that is important for a designer, and which regards to the considerations of deflections and its effects. A part may prove unsatisfactory in service as a consequence of excessive deformations, even though the stresses are in the allowable limits of the yield point. This may be caused by the *strain* that the element is supporting because of external forces.

The concept of strain has a major role in determining if an element is going to be able to support external loads, because this is a directly measurable quantity, not like stress. So, making stress analysis necessarily involves making strain analysis, because it must be first obtained experimental strain data, and then transposed into terms of stress by means of some relations.

To understand the concept of strain, we must review again the concept of elasticity. Any element consists of small particles between which forces are acting. These particles tend to avoid the deformation created by these forces by absorbing the energy, but there is a limit where particles stop absorbing energy, and as a consequence the shape of the body changes. According to Timoshenko, “under the action of external forces the particles of the body are displaced and the displacements continue until equilibrium is established between the external and internal forces” (Timoshenko, 1962). When this state is reached, it is said that the body is in a state of strain.

As said before, during the deformation, the external forces do work on the body, and this work is converted into potential energy, which is called potential energy of strain. If the external forces are then removed, the body can return to its original shape or it can be left with permanent set. In any of these two cases, some potential energy of strain may be recovered as external work.

The property by which a body returns to its original shape after removing an external force is called elasticity. As said in Chapter 2, a body is perfectly elastic if after applying an external force and removing it, it returns to its original shape. It is said to be partially elastic if after removing the external force it leaves permanent set in the body. For a designer, it is important to know the magnitude of the external loads acting on the elements, because that way the body can be designed to work as a perfectly elastic body by establishing the limits of the forces that it can support, and this must be accomplished under all service conditions.

When many forces are applied to an element, individual points of the body generally move. To be able to identify how much these points are moving, a reference plane has to be taken. The movement of a point respect to the plane taken as reference is a vector quantity known as displacement. There can be two types of displacements: linear and torsional, so displacements are associated with translation or rotation of the body. As said before, there can be two kinds of displacements: elastic and partially elastic. Elastic strains are not as important as partially elastic, because they return to its original shape. But if displacement is induced by temperature change or by an applied load, individual points of the element move relative to each other, so the size or shape of the body is changed. "The change in any dimension associated with these load or temperature induced displacements is known as a deformation…" (Riley, et. al., 1989), and displacement will be denoted as δ.

Under general conditions of loading, deformations will not be uniform throughout the whole element. As stated by Riley et. al., "Some line segments will experience extensions while other will experience contractions. Different segments…along the same line may experience different amounts of extension or contraction." (Riley et. al., 1989). But linear changes are not the only changes that a body can suffer because of applying external loads. There can be also angle changes between line segments.

So finally, after reviewing these concepts, it can be said that "strain is a quantity used to provide a measure of the intensity of a deformation (deformation per unit length)…" (Riley et. al., 1989). The classification for the strains is similar to that for the stresses. There is Normal strain (ε), used to measure the linear deformation of an arbitrary line segment, and Shearing strain (γ), which measures the angular distortion, which is the change in angle between two lines that are perpendicular before being deformed.

The strain can be produced because of temperature changes, changes in stress, or for other physical phenomena like grain growth or shrinkage, but the most common are those created by change temperatures and by stress changes. The sign convention for the strains is the same as for the stresses, and is directly related: "a positive stress tends to produce positive strain, while a negative stress tends to produce negative strain." (Juvinall, 1967).

Most of the elements that are used in machines or as engineering structures are designed to work under relatively small deformations, and this involves that the element has to behave like a perfectly elastic body, which means that it is only working on the straight-line portion of the stress-strain diagram. In this zone of the diagram, the stress and the strain are related by the Modulus of Elasticity, and they are proportional to each other. This relationship is known as Hooke's law (see Figure 9). As said by Timoshenko, "By direct experiment with the extension of prismatic bars it has been established for many structural materials that within certain limits the elongation of the bar is proportional to the tensile force." (Timoshenko, 1962). This is another way of explaining the linear relationship between the force and the elongation formulated in Hooke's law.

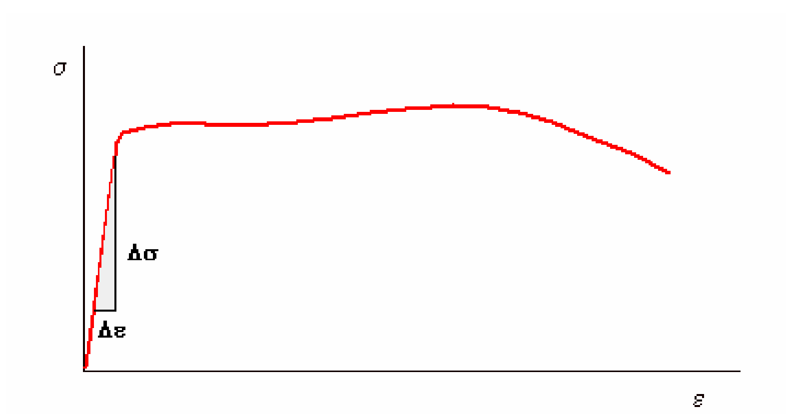


Figure 9 (Beer & Johnston et. al. Interactive Tutorial, 2001)

Hooke's law may be given by the following equation:

$$\delta = \frac{P \cdot l}{A \cdot E}$$

where P is the force producing extension of a bar, l is the length of the bar, A is the cross-sectional area of the bar, δ is the total elongation of bar, and E is the Modulus of Elasticity.

As seen in the equation, the elongation of the bar is directly proportional to the external load applied and to the length of the bar, while it is inversely proportional to the area and to the modulus of elasticity. If a tensile test is applied, it can be said that "during tension all longitudinal fibers of the prismatic bar have the same elongation and that cross sections of the bar originally plane and perpendicular to the axis of the bar remain so after extension." (Timoshenko, 1962). If we would like to calculate the elongation of the bar per unit length we could use the equation $\varepsilon = \frac{\delta}{l}$, where ε is the strain, δ is the total elongation of the bar, and $\boldsymbol{l}$ is the length of the bar. This is called the unit elongation of the tensile strain.

From the equation of tensile strain, and remembering that the formula to calculate the average normal stress is $\sigma = \frac{P}{A}$, we can express Hooke's law in another way. To make this, we have to look back to the first equation given, and making the correspondent substitutions, we can get that $\sigma = \varepsilon \cdot E$, where we can also see the linear relationship between the strain and the stress.

Hooke's law can be applied for anisotropic materials and for isotropic materials. It will be first analyzed for anisotropic materials. From Chapter 2, we know that there are six possible components of stress. As there is a direct relationship between stress and strain, there are also six components of strain. Since there are many components of stress and strain, the principle of superposition is used to make analysis easier. This principle "…asserts that the resultant stress or strain in a system subjected to several forces is the algebraic sum of their effects when applied separately." (Popov, 1968). This statement is valid only if each stress causing the strain is directly and linearly related to it, and if the strains caused by one stress component does not have a large effect on another stress. From here, it can be said that the six components of strain derived from the six components of stress are:

$$\varepsilon_x = A_{11}\tau_{xx} + A_{12}\tau_{yy} + A_{13}\tau_{zz} + A_{14}\tau_{xy} + A_{15}\tau_{yz} + A_{16}\tau_{zx}$$
$$\varepsilon_y = A_{21}\tau_{xx} + A_{22}\tau_{yy} + A_{23}\tau_{zz} + A_{24}\tau_{xy} + A_{25}\tau_{yz} + A_{26}\tau_{zx}$$
$$\varepsilon_z = A_{31}\tau_{xx} + A_{32}\tau_{yy} + A_{33}\tau_{zz} + A_{34}\tau_{xy} + A_{35}\tau_{yz} + A_{36}\tau_{zx}$$
$$\gamma_{xy}/2 = A_{41}\tau_{xx} + A_{42}\tau_{yy} + A_{43}\tau_{zz} + A_{44}\tau_{xy} + A_{45}\tau_{yz} + A_{46}\tau_{zx}$$
$$\gamma_{yz}/2 = A_{51}\tau_{xx} + A_{52}\tau_{yy} + A_{53}\tau_{zz} + A_{54}\tau_{xy} + A_{55}\tau_{yz} + A_{56}\tau_{zx}$$
$$\gamma_{zx}/2 = A_{61}\tau_{xx} + A_{62}\tau_{yy} + A_{63}\tau_{zz} + A_{64}\tau_{xy} + A_{65}\tau_{yz} + A_{66}\tau_{zx}$$

All the letters A are elastic constants, and according to these equations, there can be 36 of them, but through energy considerations it can be shown that the number of independents can be reduced to 21[1]. To be able to make this reduction, it must be assumed that the material is homogeneous. Hooke's law, in the most general form, is applicable to homogeneous anisotropic materials. Anisotropic materials are those which have "…different mechanical properties in different directions with reference to their crystallographic planes." (Popov, 1968). The general form of Hooke's law for anisotropic

[1] From Popov, Edgar*, Introduction to Mechanics of Solids*, Prentice-Hall, New Jersey, 1968, p, 100.

materials is barely used, because the shearing stresses cause a linear strain that is very small, so it can be neglected.

The equations previously discussed can be applied for isotropic materials too. We have to take into account again the principle of superposition, and to be able to achieve successfully the two conditions, we have to be sure that "…the stresses do not exceed the proportional limit for the material." (Riley et. al., 1989), which will ensure that the first condition is accomplished, and the second condition will be satisfied if "…the deformations are small, so that the small changes in the areas of the faces of the element do not produce significant changes in the stresses." (Riley et. al., 1989).

Having this in mind, the generalized Hooke's law for an isotropic material can be written as follows:

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

where E is the modulus of elasticity and ν is Poisson's ratio. It must be said that for these equations the tensile stresses and strains are taken as positive, and the compressive ones are taken as negative. These strains are calculated from the formulas of figure 10, and according to each case as seen.

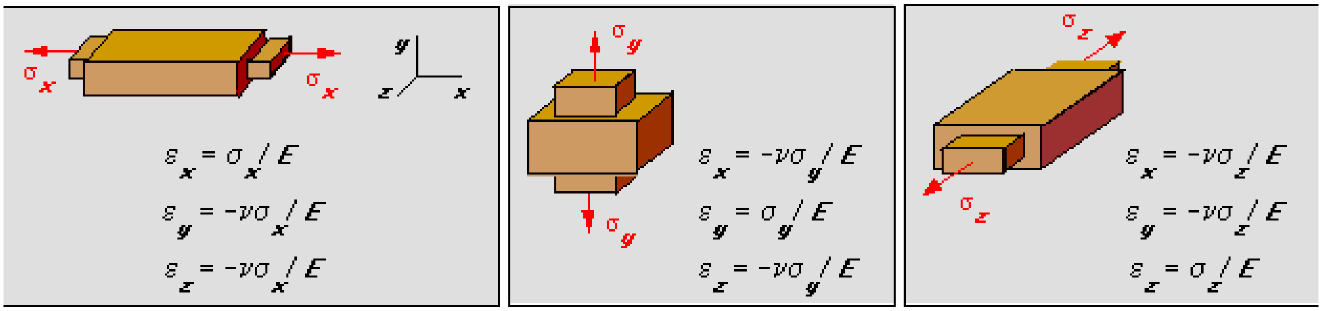


Figure 10 (Beer & Johnston et. al. Interactive Tutorial, 2001)

The normal stresses can be calculated in terms of strains with these equations derived from the ones shown upon:

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)]$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_y + \nu(\varepsilon_z + \varepsilon_x)]$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_z + \nu(\varepsilon_x + \varepsilon_y)]$$

Hooke's law can also be extended for shearing stresses, which will cause shearing strains. One single shearing stress produces only one single corresponding shearing strain, and can be expressed as:

$$\tau = G \cdot \gamma$$

where τ is the shearing stress, ***G*** is the modulus of rigidity, and γ is the shearing strain.

As seen in the past equations, there are three elastic constants: *E*, ν and *G*. These three constants can be related with the equation

$$G = \frac{E}{2(1+\nu)}$$

and with this equation, and also the equation for the shearing strain, we can get the last three components of strain:

$$\tau_{xy} = G\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

$$\tau_{yz} = G\gamma_{yz} = \frac{E}{2(1+\nu)}\gamma_{yz}$$

$$\tau_{zx} = G\gamma_{zx} = \frac{E}{2(1+\nu)}\gamma_{zx}$$

One last concept that must be understood before starting analyzing the kinds of strains is Poisson's ratio. If a solid element is subjected to axial loading, the body will contract laterally. On the other hand, if it is compressed, the element will increase sidewise, so the deformation can be easily determined just by knowing in which direction the force is acting. "The ratio of the absolute value of the strain in the lateral direction to the strain in the axial direction is Poisson's ratio." (Popov, 1968). Poisson's ratio can be expressed as $\nu = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x}$, which means lateral strain over axial strain. Because of the arguments presented before, it can be seen that this formula is only valid for strains caused by uniaxial stress.

So far, we have seen equations and explanations that are valid for general cases, but we can have more specific situations, like the one presented here in this thesis. We have now to analyze how the strain is in the shafts, the stresses it produces, how the shaft behaves, and how we can calculate the values of these stresses and strains by making the proper analysis.

When dealing with shafts, it is a frequently encountered problem that they have to transmit a torque, couple or twisting moment from one plane to a parallel plane, which produces torsional stresses. Generally, shafts have power transmission elements, like gears, pulleys, sprockets, etc., and also have bearings to avoid as much friction as possible. “The torsion problem is concerned with the determination of stresses in the shaft and deformation of the shaft.” (Riley et. al., 1989). These stresses are produced because of the changes in cross-sections to be able to place all the transmission elements, and the deformations can be caused because of the power transmission, torque, and maybe even because of the weight of the elements.

To understand the strains produced because of torsion, we have to first understand some fundamental concepts. To be able to do that, consider Figure 11. There, it can be seen a circular shaft that is attached to a wall on one end, and the

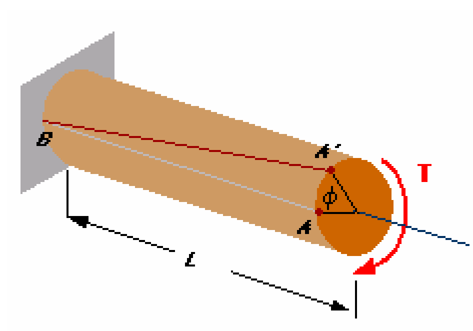
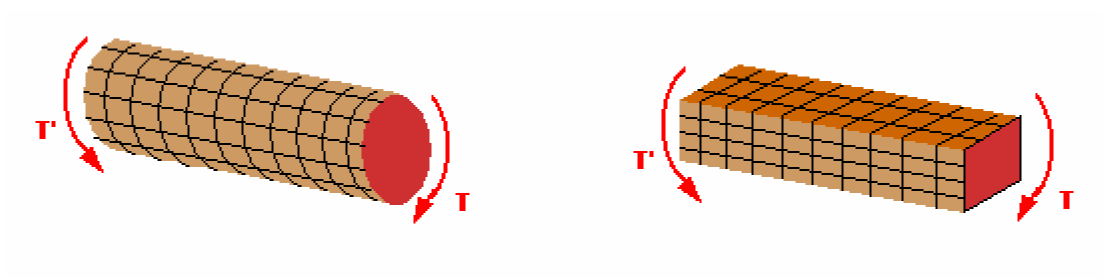


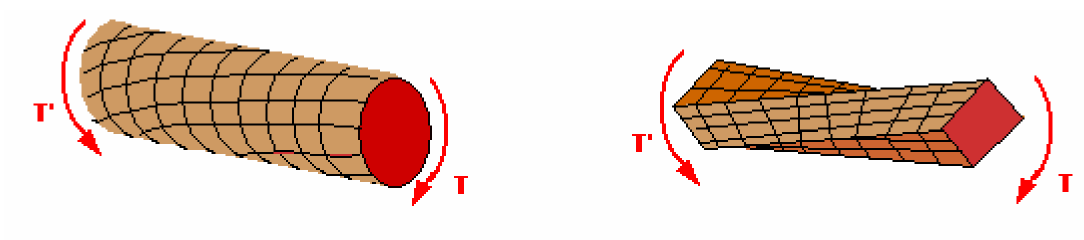
Figure 11 (Beer & Johnston et. al. Interactive Tutorial, 2001)

other end is free but a torque is being applied. As seen, after the torque is applied and maintained there, the point B remains in the same place, while point A changed its position to A’. This rotation has an angle, which is known as the angle of twist (ϕ).

Torque has a relation with the angle of twist: if we apply more torque, the angle of twist will be bigger. There can be some values of torque for which the angle of twist is proportional to the applied torque and the length of the shaft. This relationship between torque and the angle of twist was developed by C. A. Coulomb, and then confirmed by A. Duleau by making the assumption that *a plane section before twisting remains plane after twisting and a diameter remains a straight line*[2]. This assumption is only valid for solid circular and hollow shafts. If the shaft has a different shape than a circular one, then it will behave in another way, as seen in Figure 12.



(a)



(b)

Figure 12 (Beer & Johnston et. al. Interactive Tutorial, 2001)

To make the analysis of the shearing strains that are produced in a shaft because of an applied torque, the shaft must be considered to be like in Figure 11 (fixed at one

[2] From *History of Strength of Materials*, S.P. Timoshenko, McGraw-Hill, New Cork, 1953.

end and free at the other with an applied torque). The strains can be calculated at any distance from the center to the surface of the shaft. This distance is known as ρ. When the torque is applied, the strains will deform every single part of the shaft, as seen in Figure 13. From this figure, we can see that the length of the arc A-A' is Lγ, which is the same as ρA. From here, we can get the equation that describes the strain: $\gamma = \frac{\rho \cdot \phi}{L}$

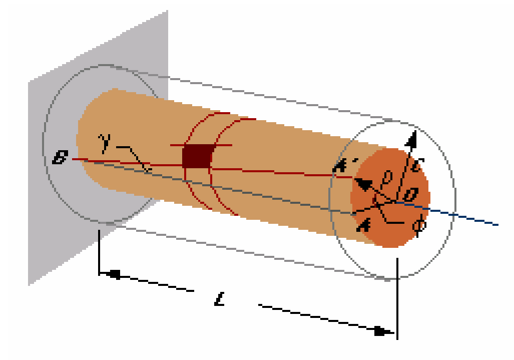


Figure 13 (Beer & Johnston et. al. Interactive Tutorial, 2001)

Here, it can be seen that the strain varies according to the radius ρ, and that it is directly proportional to the angle of twist (ϕ) and the radius (ρ), and inversely proportional to the length (L) of the shaft. The length of the shaft is constant, so with this equation can also be demonstrated that the maximum strain occurs where the radius is maximum, which is at the surface.