2. STRESSES

Two fundamental concepts in design are stress and strain. If we want to have a successful design, we have to understand these two concepts, their behavior, and their effect on every designed element. As a matter of fact, we have to design every element to support some amount of stress and strain.

2.1 Definition

To be able to understand these fundamental concepts, we have to first understand the theory of elasticity, which provides the basis for the proper stress and strain analysis. This theory provides more powerful methods to solve certain engineering problems, principally for three-dimensional elements. For example, it provides the means to investigate stresses in regions of sharp variation in cross section of shafts. As it is known, high stress concentrations occur at reentrant corners and that is the zone where cracks are likely to start. If we are able to avoid those zones, or minimize the stress effect, we will have a more efficient element, which will last more, and will have a longer service life.

When designing an element, we always want to work on the elastic zone. As said by Timoshenko, “the fundamental problem for the designer is to establish the proportions of the members of the structure such that it will approach the condition of a perfectly elastic body under all service conditions” (Timoshenko, 1962). This is because we do not want our element to suffer permanent deformation, since this will affect its performance and its service life, and what we want to achieve is an element with the best characteristics to last more.

STRESS-STRAIN CURVE

To understand the concept of elastic zone, we must first take a look at the stress-strain diagram. The one presented here is for typical structural steel in tension, and it is one of the most typical curves, even if there are other kinds. The point from 1 to 2 represents the zone where the relationship between stress and

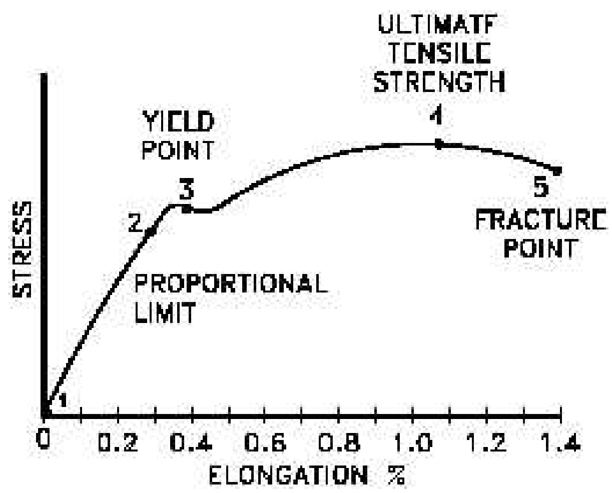

Figure 1

strain is proportional, and that is why the point 2 is called the proportional limit. The slope of the straight line from 1 to 2 is called the modulus of elasticity.

After passing the proportional limit, the stress keeps increasing, as it does the strain, but strain increases more rapidly. As a consequence of this, the curve and the slope decreases until reaching a point where it becomes almost horizontal, and then the slope becomes negative until reaching point 3. At this point, considerable elongation occurs with no noticeable increase in the tensile force, phenomenon known as yielding. So point 3 is known as yield point. The stress at this point is known as the yield stress.

From point 3 to point 4 of the curve, the material starts to suffer some changes in its crystalline structure, causing an effect known as strain hardening. This allows the material to increase its resistance for further deformation. At point 4, the load reaches its maximum value, and that is why it is called the ultimate tensile strength. By strength, it must be understood as the capacity of a structure to resist loads (Gere & Timoshenko, 1997). This can be taken also as the failure load. The corresponding stress for this point is called ultimate stress. After point 4, the load decreases until reaching point 5, where fracture occurs.

ELASTIC AND PLASTIC BEHAVIOR

We can divide the stress-strain curve in two parts: the part where the element behaves elastically and the part where it behaves plastically. Take in

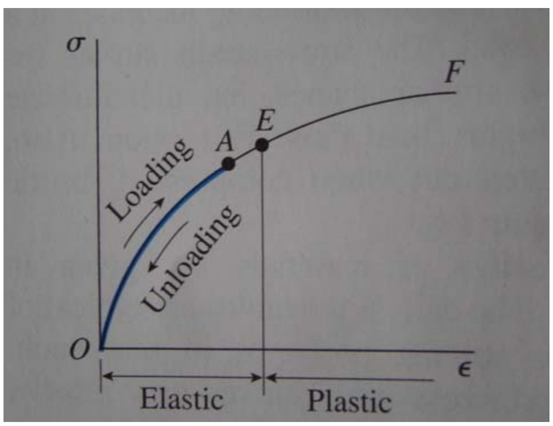

Figure 2 (source G & T, 1997)

consideration Figure 2. If we apply the load such that it goes from point O to point A and then remove it, and if the curve follows the same path, then it means that it is working in the elastic zone, and that no permanent deformation occurs in the element. If, on the other hand, we apply the load until passing the yield point

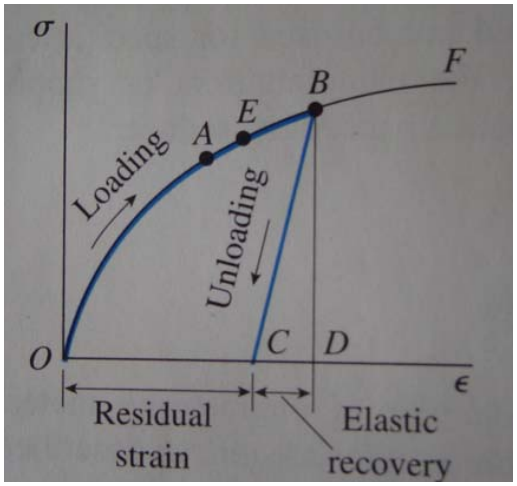

Figure 3 (source G & T, 1997)

(Figure 3), and then remove it, we are going to be working on the plastic zone and the element will suffer permanent set. There is also residual strain, which makes the element longer than it was before. As seen, it does not follow the same returning path of the stress-strain curve, but it returns to a point different than zero. If we apply the load again, the curve may follow the same path of the stress-strain curve, but it starts from point C until reaching point F (Figure 3). If this happens, strain continues increasing, while stress increase is little until rupture happens. So, as stated before, what a designer wants is to ensure that the material is in the elastic zone during its working life, so that way it will work as required under certain conditions without suffering any permanent set.

To avoid permanent set, most engineering elements are designed to work under relatively small deformations, involving only a part of the straight line of the stress-strain curve, where the stress and the strain are proportional. From this, and knowing that the Young's modulus is the slope of the straight line, we can get into a relationship known as Hooke's law, where

$$\underline{\sigma = E\varepsilon}$$

This is a relation independent of the direction of loading, and this is because the mechanical properties are independent of the direction considered. Such materials are known as isotropic. If the material properties depend upon the direction of the loading, those materials are said to be anisotropic.

FATIGUE

Until now, we have only considered elements that are subjected to an axial loading, and we can say that if the maximum stress in the element does not exceed the elastic limit of the material, it will return to its original shape when the load is removed. But talking about loadings that are repeated constantly over a certain period of time, the behavior is not the same. In this case, “rupture will occur at a stress much lower than the static breaking strength” (Beer et. al., 2001), phenomenon known as fatigue, and “a fatigue failure is of a brittle nature, even for materials that are normally ductile” (Beer & Johnston et. al., 2001). This is the type of analysis that should be considered for shafts, because shafts are subjected to fatigue, which is a case of tension-compression stresses. In one moment of the cycle, one part of the shaft is subjected to tension, while the other is subjected to compression. After half cycle, the part that was subjected to tension will be now subjected to compression, and vice versa. These two situations are present for every point of the shaft in one cycle, so the shaft is subjected to tension and compression stresses all its service life, letting the formation of cracks, and after a while, the failure of the element.

To make the fatigue analysis, we need to know the number of loading cycles that an element can support. The element has to be subjected to repeated successive loadings and reverse loadings, and this will give a specific maximum stress level. To understand better how the element behaves we can plot the results in a stress-cycles curve (□-*n* curve) (Figure 4). As seen, when the stress value is

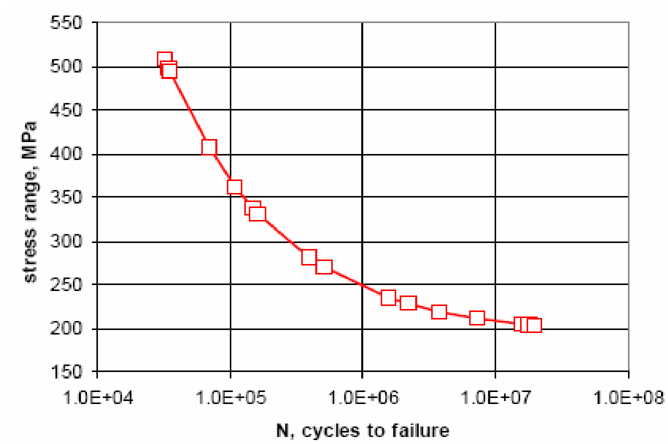

Figure 4

high, the number of cycles that the element can support is low. And as the value of the stress decreases, the number of cycles increases, so service life increases also. It is an inversely proportional relationship. When the magnitude of the stress is reduced and as a consequence the cycles to cause rupture increases, the element will reach a stress known as endurance limit. "The endurance limit is the stress for which failure does not occur, even for an indefinitely large number of loading cycles.

According to experiments, it is known that failures start from microscopic cracks or similar imperfections. As the cycles keep running, at each cycle the crack is slightly enlarged until reaching a point where the undamaged material is not capable of carrying the maximum load, and then a sudden brittle fracture occurs. So, as a way of increasing the number of cycles that an element can support, the surface condition of it must be kept good, trying to avoid corrosion, and similar conditions that make the element more prone to failure.

TYPES OF STRESSES

So far, it has only been talked about the theory of elasticity and how it helps to solve engineering problems easily, mostly those involving three-dimensional elements. It has also been talked about the fundamental concepts to understand that a designer has to work on the elastic zone of a material in order to maintain it working adequately, and also of the definition of fatigue, and how can it affect the performance of an element. But now, a deeper analysis has to be made regarding to the types of stresses that are present in an element, and the influence that they have on the performance and on the service life of it.

Taking a body in equilibrium, and applying some external forces, internal forces will be produced between the parts of the body. If a cut is made through this body, it can be said that the distribution of the internal forces over the cross section area is uniform. The magnitudes of such forces are usually defined by their intensity, so it can be said that the amount of force per unit area of the surface is an intensity called stress. If we were talking about an inclined plane, then we will have two components of the stress: a normal stress, which is perpendicular to the cross section area, and a shearing stress which acts in the plane of the area.

There are two kinds of external forces which may act on bodies: surface forces, which are distributed over the surface of the body; and body forces, which are distributed over the volume of a body (Timoshenko et. al., 1970). If we take a little part of the body, we could get a small cubic element at one specific point with sides parallel to the coordinate axes. By denoting the normal stresses as σ and the shearing stresses as τ, we can see how they act on each of the sides of the cubic element. For an easier analysis, we are going to assume that all the stresses are positive. So, making a representation of this (Figure 5), we will get that each of the sides has a normal stress perpendicular to the plane and two

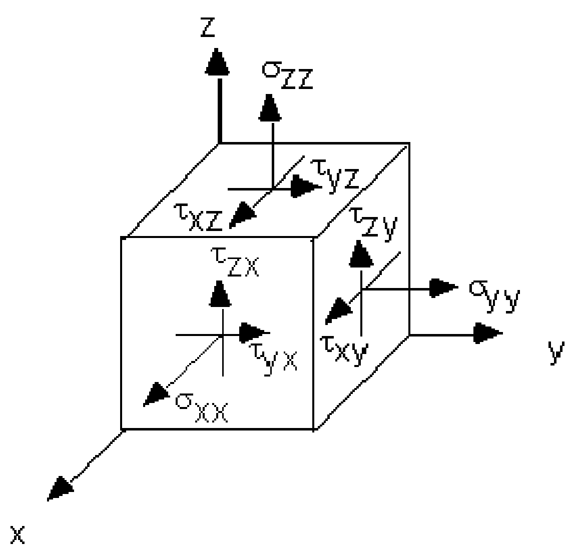

Figure 5

shearing stresses acting on the plane. For the normal stresses, the subscript letters indicate that the stress is acting on a surface normal to the one that appears on the subscript. For the shearing stresses, the first subscript letter indicates the direction of the normal to the plane, and the second subscript letter indicates the direction of the component of the stress. Then, we will have six normal stresses (denoted only by three symbols: σ_x, σ_y, σ_z)

and twelve shearing stresses (denote only by six symbols: τ_{xy}, τ_{yx}, τ_{xz}, τ_{zx}, τ_{zy} and τ_{yz}). But making a consideration of the equilibrium in the cubic element, the shearing stresses can be reduced to three. By taking the moments of the element in a line that passes through the center of the cube, it can be seen that $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$ and $\tau_{zy} = \tau_{yz}$. Hence, "for a two perpendicular sides of a cubic element the components of shearing stress perpendicular to the line of intersection of these sides are equal" (Timoshenko et. al., 1970). These, plus the three components of normal stress are known as the components of stress at a point.

- **Loads**

But to first understand the type of stresses, it must be first understood the types of loads that can cause them. If loads are classified with respect to time, there are static, sustained, impact and repeated loads. When talking about a static load, we are referring to a load that is applied gradually, and that eventually will reach equilibrium in a short period of time. A sustained load is the same as a static load, but applied constant over a long period of time. An impact load is a suddenly applied load, which may produce vibration, and it will reach equilibrium after the body stops vibrating. Finally, "A repeated load is a load that it is applied and removed many times" (Riley et. al., 1989), like that presented in fatigue as explained before.

Classifying loads with respect to the area over where they are applied, loads may be classified in concentrated and distributed. A concentrated load is a load that it is applied only at a specific point. In the other hand, a distributed load is a load which is distributed not only at a specific point, but over a length or a section of the body. The distribution can be uniform or not, and it can be converted to a concentrated load.

The last classification that can be made for loads is depending on the location and on the method of application. There are four kinds: centric, torsional, bending and combined loads. A centric load, as its name says, is a load which passes through the centroid of the section where the load is being applied. If the force is applied on the centroid of all the sections of the element, then it is said to be axial. A torsional load is a force that subjects an element to couples that twist the member, like shafts. A bending load, also known as flexural, is the one where the loads are applied transversely to the longitudinal axis of the member; for example, in one single member there can be applied concentrated, uniformly distributed loads and also a moment. Finally, a combined loading is a combination of two or more types of loading.

After analyzing the types of loads that can be present in an element, now we can talk about the types of stresses that can be created when applying these loads. Basically, we can find four kinds: normal stresses (which are also known as tension or compression stresses), shearing stresses, bending stresses and torsional stresses.

2.2 Normal stresses

There can be a lot of external forces acting over a body, but to simplify the analysis, we can assume as if we were concentrating an average of all these forces in a single force acting on the centroid of the body. Now, normal stress can be defined.

Normal stress is the intensity of the force acting normal to an infinitesimal area *dA* within an object per unit area. As said, the force has to be acting on the centroid. If the force is pulling the body, then it generates a tensile stress, while if the force is pushing the body, then it generates a compressive stress. An average stress can be calculated from the formula $\sigma = \frac{F}{A}$ where F represents the force and A the area of the cross section. It can be expressed in psi's or in Pascals. As said before, this formula gives an average value, and taking into account Saint Venant's Principle which states that "…except in the immediate vicinity of the points of application of the loads, the stress distribution may be assumed independent of the actual mode of application of the loads." (Beer & Johnston et. al., 2001), it can be assumed that the normal stresses distribution on an element with an axial load is uniform, except in the vicinity of where the load is applied.

2.3 Shearing stresses

If instead of applying the force axially we apply the force perpendicular to the axis, we will obtain very different kind of forces, so the stresses will be different also. What we are doing here is that instead of creating forces normal to the cross section area, we are creating forces on the plane of the section. These forces are known as shearing forces. Similarly as in normal stress, here we can also find an average shearing stress with the formula $\tau_{ave} = \frac{V}{A}$, where V denotes the shearing force. As mentioned before, it must be understood that the value obtained from this equation is an average value, and that, contrary as what we did in normal stresses, here the stress distribution cannot be taken as uniform. As mentioned by Beer & Johnston et. al., "…the actual value of τ of the shearing stress varies from zero at the surface of the member to a maximum value τ_{max} that may be much larger than the average value τ_{ave}." (Beer & Johnston et. al., 2001).

If we are talking about a method where the loads are transferred from one member to another by bolts or pins, then there can arise two types of shear: single and double shear. Single shear is when we are using only one cross section of the bolt to transfer the load, and here the shearing force V is equal to the applied force P. Double shear is when we are using both parts of the cross section to transmit the load between the members, and here, the resultant shearing force V is equal to one-half of the applied force P.

- **Bearing stresses**

When talking about shearing stresses, we may include also the bearing stresses. This is considered also for elements that transfer the load from one to another. For example, if we have a plate and a clevis, and they are kept together by a bolt, the bolt will develop contact stresses (also known as bearing stresses). Additional to this, the plate and the clevis are also creating shearing stresses. As bearing stresses distribution is difficult to obtain, "…it is customary to assume that the stresses are uniformly distributed." (Gere & Timoshenko, 1997). So, assuming that the stress distribution is constant, then the bearing stress can be calculated as $\sigma_b = \frac{F_b}{A_b}$ where F_b and A_b are the bearing force and area respectively, where the bearing area is the projected area of the curved bearing surface.

- **Bending stresses**

Referring to bending stresses, they are applied mostly for beams. Bending stress is the normal stress that is induced at a point in a body subjected to loads that cause it to bend. If a load is applied perpendicular to the length of the beam, and if the beam has two supports on each end, then bending moments are going to be induced in the beam. If a beam is supported in the ends, and a load is applied in the middle point of it, it will suffer bending. When the beam is in bending (if a load is applied on the top of the beam), the top of the beam will be experiencing maximum compression, while the bottom will be experiencing maximum tension.

To calculate the stresses in terms of the bending moment, we can use the formula $\sigma_x = -\frac{My}{I}$. This equation is also known as the flexure formula, and it can be seen that "the stress is directly proportional to the bending moment M and inversely proportional to the moment of inertia I of the cross section" (Gere & Timoshenko, 1997). Also, the value of the stress is going to vary directly with the distance *y* from the neutral axis.

2.4 Torsional stresses

When making evaluations for beam stresses and deflections, it must be also considered the impact of any torsional loads. There are two types of torsional loads: rotation and warping. For rotation, a member undergoing torsion will rotate about its shear center through an angle of f as measured from each end of the member. This rotational displacement function ϕ and its derivatives with respect to member length are used to determine the torsional stresses of the member. For warping, torsion on a member will result in the cross section rotating a given amount. Non-circular sections will also experience warping of the cross section. In addition to circular cross sections, this warping will not occur on sections where the section is composed of plates and the centerline of the members forming the shape meet at a common point. The warping stresses of the member are dependent upon any restraint of the cross section's ability to deflect.

The stresses induced in an element by torsion can be classified in three: torsional shear stress, warping shear stress and warping normal stress. Pure torsional shearing stresses are those which act in a direction parallel to the edges of the particular shape's element. The variation of these stresses is linear across the thickness of the element. No matter which the shape is, the maximum stress is

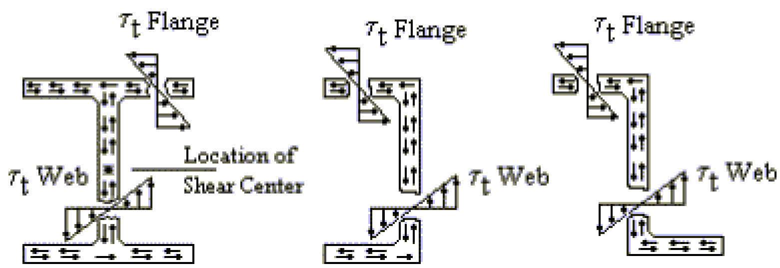

Figure 6

going to be present in the thickest element, as seen in Figure 6. The maximum shearing torsional stress for the cross section can be determined from the equation

$\tau_t = Gt\phi$, where *G* is the shear modulus (or modulus of Rigidity), *t* is the thickness of the element and ϕ is the first derivative of the rotational displacement function.

Warping stresses are produced when the member is restrained in a way that the cross section cannot warp freely. This can generate normal and also shearing warping stresses. The shearing stresses act in a direction that is parallel to the edges of the particular shape's element. The distribution of the stresses is constant across the thickness of the element, but they vary with the length. To calculate the warping shear stress, the formula is $\tau_w = \frac{-ES_w\phi''}{t}$, where E is the Young's modulus (or Elasticity modulus), S_w is the warping statical moment at a point on the cross section, t is the thickness of the element and ϕ'' is the second derivative of the rotational displacement function. The distribution of the stresses of this kind of stress is as seen in Figure 7.

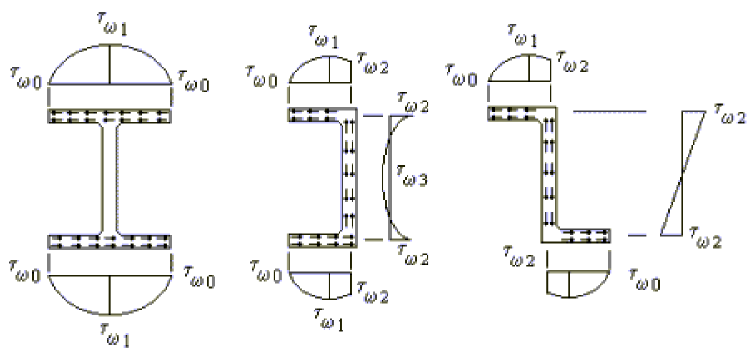

Figure 7

In what refers to warping normal stresses, these are stresses that result from bending of the element due to torsion, and as all the normal stresses we have seen so far, they can be tensional or compressive. These stresses act perpendicular to the surface of the cross section. As shearing stresses, they are constant along the thickness of the element, and non uniform along the length. These stresses can be calculated from the formula $\tau_w = EW_{ns}\phi'''$, where E is the Young's modulus, W_{ns} the normalized warping constant at a point on the cross section and ϕ''' is the third derivative of the rotational displacement function. To see how these normal stresses act on an element, see Figure 8.

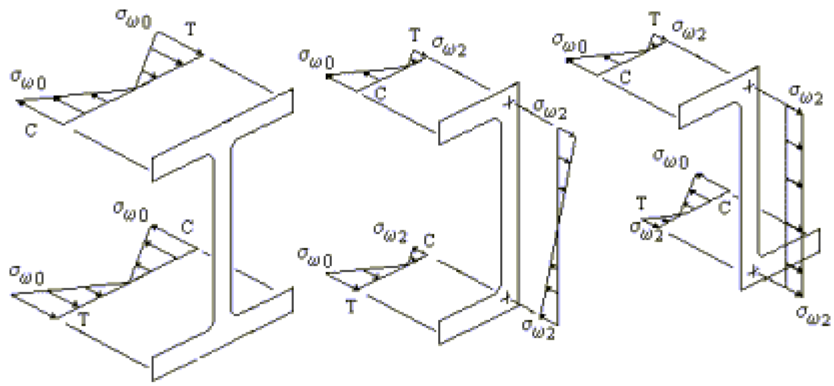

Figure 8

Referring to circular shafts, and the torsional stresses produced in them, it can be stated that "…the shearing stresses produced on any transverse plane are always perpendicular to the radius to the point." (Riley et. al., 1989). The French engineer C. A. Coulomb developed an analytical relationship in the one he stated that *a plane section before twisting remains plane after twisting and a diameter remains a straight line*. This is only valid for solid or hollow circular shafts, because before and after applying a torque its cross section remains circular.

If additional to this assumption, we say that the shaft is working according to Hooke's law, which means that the shaft is working in a safe elastic zone under the proportional limit, the shearing stress for torsion can be calculated from $Tr = \frac{\tau_p J}{\rho}$, or which is the same $\tau_p = \frac{T\rho}{J}$, where T is the applied torque, ρ is the distance from the axis of the shaft to the point where we want to calculate the stress, and J is the polar moment of inertia. This equation is known as the elastic torsion formula, and is only valid for linearly elastic action in homogeneous and isotropic materials. The maximum shearing stress will occur at the outside surface of the shaft, and that is why we take that ρ is equal to the radius of the shaft.