Capítulo 4

Dinámica Bosónica No-Conmutativa y Fermiónica No-Anticonmutativa

En esta sección abordaremos el caso en el cual tenemos un sistema que tiene N grados de libertad bosónicos y N grados de libertad fermiónicos (caso No Supersimétrico). Aclaremos que el caso más general sería aquel en el cual se tiene un sistema con N grados de libertad bosónicos y M grados de libertad fermiónicos (caso Supersimétrico M=N), no obstante, este caso sale del contexto de este trabajo de tesis (debido a las herramientas matemáticas que requiere) y no se abordará.

Partamos de una estructura simpléctica con la siguiente forma:

$$\begin{pmatrix} \begin{pmatrix} \{x_i, x_j\}_P & \{x_i, p_j\}_P \\ \{p_i, x_j\}_P & \{p_i, p_j\}_P \end{pmatrix} & \mathbf{0}_{2N\times 2N} \\ \mathbf{0}_{2N\times 2N} & \begin{pmatrix} \{\eta_i, \eta_j\}_P & \{\eta_i, \eta_j^*\}_P \\ \{\eta_i^*, \eta_j\}_P & \{\eta_i^*, \eta_j^*\}_P \end{pmatrix} \end{pmatrix}. \tag{4.1}$$

La cual, es el caso particular de:

$$\begin{pmatrix} \begin{pmatrix} \{x_i, x_j\}_P & \{x_i, p_j\}_P \\ \{p_i, x_j\}_P & \{p_i, p_j\}_P \end{pmatrix} & \begin{pmatrix} \{x_i, \eta_j\}_P & \{x_i, \eta_j^*\}_P \\ \{p_i, \eta_j\}_P & \{p_i, \eta_j^*\}_P \end{pmatrix} \\ \begin{pmatrix} \{\eta_i, x_j\}_P & \{\eta_i, p_j\}_P \\ \{\eta_i^*, x_j\}_P & \{\eta_i^*, p_j\}_P \end{pmatrix} & \begin{pmatrix} \{\eta_i, \eta_j\}_P & \{\eta_i, \eta_j^*\}_P \\ \{\eta_i^*, \eta_j\}_P & \{\eta_i^*, \eta_j^*\}_P \end{pmatrix} \end{pmatrix}, \tag{4.2}$$

es decir, estamos considerando que en el sistema no hay combinación entre las variables bosónicas y fermiónicas, esto es, son independientes entre sí dichas variables.

Para tomar los casos No-conmutativos y No-anticonmutativos, usaremos los siguientes generadores: Para la parte de las variables bosónicas:

$$\begin{aligned}\{x_\alpha, x_\beta\}_P &= \theta_{\alpha\beta},\\ \{x_\alpha, p_\beta\}_P &= \delta_{\alpha\beta},\\ \{p_\alpha, p_\beta\}_P &= \beta_{\alpha\beta}.\end{aligned}$$

y para la parte de las variables fermiónicas.

$$\begin{aligned}\{\eta_\alpha, \eta_\beta\}_P &= \varphi_{\alpha\beta},\\ \{\eta_\alpha, \eta^*_\beta\}_P &= -2i\delta_{\alpha\beta},\\ \{\eta^*_\alpha, \eta^*_\beta\}_P &= \vartheta_{\alpha\beta}.\end{aligned}$$

Ya teniendo definida la estructura simpléctica a utilizar, tomemos el Hamiltoniano completo de $\mathbf{G}_2$ (es decir incluyendo grados de libertad bosónicos y fermiónicos):

$$H = \frac{p_\beta^2}{2m} + V_1(\mathbf{x}) + \eta^*_\beta \eta_\beta V_2(\mathbf{x}), \tag{4.3}$$

y calculemos las ecuaciones de movimiento correspondientes, empecemos con x_α:

$$\dot{x}_\alpha \ = \{x_\alpha, H\}_P \ = \{x_\alpha, \tfrac{p_\beta^2}{2m} + V_1(\mathbf{x}) + \eta^*_\beta \eta_\beta V_2(\mathbf{x})\}_P.$$

Aquí hay que tener presente las definiciones para los paréntesis de Poisson dadas en (2.21, 2.22, 2.23); aumentándoles los términos de corrección. Tanto para x como para p se debe usar la definición:

$$\begin{aligned}\{E_1, E_2\}_P &= +\theta_{ij}\tfrac{\partial E_1}{\partial x_i}\tfrac{\partial E_2}{\partial x_j} + (\tfrac{\partial E_1}{\partial x_i}\tfrac{\partial E_2}{\partial p_i} - \tfrac{\partial E_1}{\partial p_i}\tfrac{\partial E_2}{\partial x_i}) + \beta_{ij}\tfrac{\partial E_1}{\partial p_i}\tfrac{\partial E_2}{\partial p_j}\\ &\quad +\varphi_{ij}\tfrac{\partial E_1}{\partial \eta_i}\tfrac{\partial E_2}{\partial \eta_j} - 2i(\tfrac{\partial E_1}{\partial \eta_i}\tfrac{\partial E_2}{\partial \eta^*_i} - \tfrac{\partial E_1}{\partial \eta^*_i}\tfrac{\partial E_2}{\partial \eta_i}) + \vartheta_{ij}\tfrac{\partial E_1}{\partial \eta^*_i}\tfrac{\partial E_2}{\partial \eta^*_j}\end{aligned} \tag{4.4}$$

puesto que x, p y H son funciones pares respecto a las variables fermiónicas

$$\begin{aligned}\dot{x}_\alpha &= \{x_i, x_j\}_P\tfrac{\partial x_\alpha}{\partial x_i}\tfrac{\partial H}{\partial x_j} + \{x_i, p_j\}_P\tfrac{\partial x_\alpha}{\partial x_i}\tfrac{\partial H}{\partial p_j}\\ &\quad +\{p_i, x_j\}_P\tfrac{\partial x_\alpha}{\partial p_i}\tfrac{\partial H}{\partial x_j} + \{p_i, p_j\}_P\tfrac{\partial x_\alpha}{\partial p_i}\tfrac{\partial H}{\partial p_j}\\ &\quad +\{\eta_i, \eta_j\}_P\tfrac{\partial x_\alpha}{\partial \eta_i}\tfrac{\partial H}{\partial \eta_j} + \{\eta_i, \eta^*_j\}_P\tfrac{\partial x_\alpha}{\partial \eta_i}\tfrac{\partial H}{\partial \eta^*_j}\\ &\quad +\{\eta_i, \eta^*_j\}_P\tfrac{\partial x_\alpha}{\partial \eta^*_i}\tfrac{\partial H}{\partial \eta_j} + \{\eta^*_i, \eta^*_j\}_P\tfrac{\partial x_\alpha}{\partial \eta^*_i}\tfrac{\partial H}{\partial \eta^*_j}\end{aligned}$$

$$\begin{aligned}
\dot{x}_\alpha &= \theta_{ij}\delta_{\alpha i}[\tfrac{\partial V_1}{\partial x_j} + \eta^*_\beta\eta_\beta\tfrac{\partial V_2}{\partial x_j}] + \delta_{ij}\delta\alpha i\tfrac{\partial\frac{p^2}{2m}}{\partial p_j} \\
&\quad -\delta_{ij}(0)[\tfrac{\partial V_1}{\partial x_j} + \eta^*_\beta\eta_\beta\tfrac{\partial V_2}{\partial x_j}] + \beta_{ij}(0)\tfrac{\partial\frac{p^2}{2m}}{\partial p_j} \\
&\quad +\varphi_{ij}(0)(-\eta^*_\beta\delta_{\beta j}V_2) - 2i\delta_{ij}(0)(-\delta_{\beta j}\eta_\beta V_2) \\
&\quad -2i\delta_{ij}(0)(-\delta_{\beta j}\eta^*_\beta V_2) + \vartheta_{ij}(0)(-\delta_{\beta j}\eta_\beta V_2)
\end{aligned}$$

$$\dot{x}_\alpha = \theta_{\alpha j}[\frac{\partial V_1}{\partial x_j} + \eta^*_\beta\eta_\beta\frac{\partial V_2}{\partial x_j}] + (\frac{p_\alpha}{m}). \tag{4.5}$$

Para $\dot{p}_\alpha$:

$$\begin{aligned}
\dot{p}_\alpha &= \{x_i, x_j\}_P\tfrac{\partial p_\alpha}{\partial x_i}\tfrac{\partial H}{\partial x_j} + \{x_i, p_j\}_P\tfrac{\partial p_\alpha}{\partial x_i}\tfrac{\partial H}{\partial p_j} \\
&\quad +\{p_i, x_j\}_P\tfrac{\partial p_\alpha}{\partial p_i}\tfrac{\partial H}{\partial x_j} + \{p_i, p_j\}_P\tfrac{\partial p_\alpha}{\partial p_i}\tfrac{\partial H}{\partial p_j} \\
&\quad +\{\eta_i, \eta_j\}_P\tfrac{\partial p_\alpha}{\partial \eta_i}\tfrac{\partial H}{\partial \eta_j} + \{\eta_i, \eta^*_j\}_P\tfrac{\partial p_\alpha}{\partial \eta_i}\tfrac{\partial H}{\partial \eta^*_j} \\
&\quad +\{\eta_i, \eta^*_j\}_P\tfrac{\partial p_\alpha}{\partial \eta^*_i}\tfrac{\partial H}{\partial \eta_j} + \{\eta^*_i, \eta^*_j\}_P\tfrac{\partial p_\alpha}{\partial \eta^*_i}\tfrac{\partial H}{\partial \eta^*_j} \\
\dot{p}_\alpha &= +\theta_{ij}(0)[\tfrac{\partial V_1}{\partial x_j} + \eta^*_\beta\eta_\beta\tfrac{\partial V_2}{\partial x_j}] + \delta_{ij}(0)\tfrac{\partial\frac{p^2}{2m}}{\partial p_j} \\
&\quad -\delta_{ij}\delta_{\alpha i}[\tfrac{\partial V_1}{\partial x_j} + \eta^*_\beta\eta_\beta\tfrac{\partial V_2}{\partial x_j}] + \beta_{ij}(\delta_{\alpha i})\tfrac{\partial\frac{p^2}{2m}}{\partial p_j} \\
&\quad +\varphi_{ij}(0)(-\eta^*_\beta\delta_{\beta j}V_2) - 2i\delta_{ij}(0)(-\delta_{\beta j}\eta_\beta V_2) \\
&\quad -2i\delta_{ij}(0)(-\delta_{\beta j}\eta^*_\beta V_2) + \vartheta_{ij}(0)(-\delta_{\beta j}\eta_\beta V_2)
\end{aligned}$$

$$\dot{p}_\alpha = -[\frac{\partial V_1}{\partial x_\alpha} + \eta^*_\beta\eta_\beta\frac{\partial V_2}{\partial \alpha}] + \beta_{\alpha j}(\frac{p_j}{m}) \tag{4.6}$$

Por su parte, para calcular $\dot{\eta}_\alpha$ y $\dot{\eta}^*_\alpha$ debemos usar la definición del Paréntesis de Poisson:

$$\begin{aligned}
\{O, E\}_P &= +\theta_{ij}\tfrac{\partial O}{\partial x_i}\tfrac{\partial E}{\partial x_j} + (\tfrac{\partial O}{\partial x_i}\tfrac{\partial E}{\partial p_i} - \tfrac{\partial O}{\partial p_i}\tfrac{\partial E}{\partial x_i}) + \beta_{ij}\tfrac{\partial O}{\partial p_i}\tfrac{\partial E}{\partial p_j} \\
&\quad -\varphi_{ij}\tfrac{\partial O}{\partial \eta_i}\tfrac{\partial E}{\partial \eta_j} + 2i(\tfrac{\partial O}{\partial \eta_i}\tfrac{\partial E}{\partial \eta^*_i} + \tfrac{\partial O}{\partial \eta^*_i}\tfrac{\partial E}{\partial \eta_i}) - \vartheta_{ij}\tfrac{\partial O}{\partial \eta^*_i}\tfrac{\partial E}{\partial \eta^*_j}
\end{aligned} \tag{4.7}$$

puesto que η y η^* son funciones impares respecto a las variables fermiónicas y H es función par.

$$
\begin{aligned}
\dot{\eta}_\alpha &= \{x_i, x_j\}_P \tfrac{\partial \eta_\alpha}{\partial x_i}\tfrac{\partial H}{\partial x_j} + \{x_i, p_j\}_P \tfrac{\partial \eta_\alpha}{\partial x_i}\tfrac{\partial H}{\partial p_j} \\
&\quad + \{p_i, x_j\}_P \tfrac{\partial \eta_\alpha}{\partial p_i}\tfrac{\partial H}{\partial x_j} + \{p_i, p_j\}_P \tfrac{\partial \eta_\alpha}{\partial p_i}\tfrac{\partial H}{\partial p_j} \\
&\quad + \{\eta_i, \eta_j\}_P \tfrac{\partial \eta_\alpha}{\partial \eta_i}\tfrac{\partial H}{\partial \eta_j} + \{\eta_i, \eta_j^*\}_P \tfrac{\partial \eta_\alpha}{\partial \eta_i}\tfrac{\partial H}{\partial \eta_j^*} \\
&\quad + \{\eta_i, \eta_j^*\}_P \tfrac{\partial \eta_\alpha}{\partial \eta_i^*}\tfrac{\partial H}{\partial \eta_j} + \{\eta_i^*, \eta_j^*\}_P \tfrac{\partial \eta_\alpha}{\partial \eta_i^*}\tfrac{\partial H}{\partial \eta_j^*} \\
\dot{\eta}_\alpha &= \theta_{ij}(0)[\tfrac{\partial V_1}{\partial x_j} + \eta_\beta^* \eta_\beta \tfrac{\partial V_2}{\partial x_j}] + \delta_{ij}(0)\tfrac{\partial \frac{p^2}{2m}}{\partial p_j} \\
&\quad - \delta_{ij}(0)[\tfrac{\partial V_1}{\partial x_j} + \eta_\beta^* \eta_\beta \tfrac{\partial V_2}{\partial x_j}] + \beta_{ij}(0)\tfrac{\partial \frac{p^2}{2m}}{\partial p_j} \\
&\quad - \varphi_{ij}(\delta_{\alpha i})(-\eta_\beta^* \delta_{\beta j} V_2) + 2i\delta_{ij}(\delta_{\alpha i})(-\delta_{\beta j}\eta_\beta V_2) \\
&\quad - 2i\delta_{ij}(0)(-\delta_{\beta j}\eta_\beta^* V_2) + \vartheta_{ij}(0)(-\delta_{\beta j}\eta_\beta V_2)
\end{aligned}
$$

$$
\dot{\eta}_\alpha = \varphi_{\alpha\beta}\eta_\beta^* V_2 - 2i\eta_\alpha V_2. \tag{4.8}
$$

Para $\dot{\eta}_\alpha^*$:

$$
\begin{aligned}
\dot{\eta}_\alpha^* &= \{x_i, x_j\}_P \tfrac{\partial \eta_\alpha^*}{\partial x_i}\tfrac{\partial H}{\partial x_j} + \{x_i, p_j\}_P \tfrac{\partial \eta_\alpha^*}{\partial x_i}\tfrac{\partial H}{\partial p_j} \\
&\quad + \{p_i, x_j\}_P \tfrac{\partial \eta_\alpha^*}{\partial p_i}\tfrac{\partial H}{\partial x_j} + \{p_i, p_j\}_P \tfrac{\partial \eta_\alpha^*}{\partial p_i}\tfrac{\partial H}{\partial p_j} \\
&\quad + \{\eta_i, \eta_j\}_P \tfrac{\partial \eta_\alpha^*}{\partial \eta_i}\tfrac{\partial H}{\partial \eta_j} + \{\eta_i, \eta_j^*\}_P \tfrac{\partial \eta_\alpha^*}{\partial \eta_i}\tfrac{\partial H}{\partial \eta_j^*} \\
&\quad + \{\eta_i, \eta_j^*\}_P \tfrac{\partial \eta_\alpha^*}{\partial \eta_i^*}\tfrac{\partial H}{\partial \eta_j} + \{\eta_i^*, \eta_j^*\}_P \tfrac{\partial \eta_\alpha^*}{\partial \eta_i^*}\tfrac{\partial H}{\partial \eta_j^*} \\
\dot{\eta}_\alpha^* &= \theta_{ij}(0)[\tfrac{\partial V_1}{\partial x_j} + \eta_\beta^* \eta_\beta \tfrac{\partial V_2}{\partial x_j}] + \delta_{ij}(0)\tfrac{\partial \frac{p^2}{2m}}{\partial p_j} \\
&\quad - \delta_{ij}(0)[\tfrac{\partial V_1}{\partial x_j} + \eta_\beta^* \eta_\beta \tfrac{\partial V_2}{\partial x_j}] + \beta_{ij}(0)\tfrac{\partial \frac{p^2}{2m}}{\partial p_j} \\
&\quad - \varphi_{ij}(0)(-\eta_\beta^* \delta_{\beta j} V_2) + 2i\delta_{ij}(0)(-\delta_{\beta j}\eta_\beta V_2) \\
&\quad + 2i\delta_{ij}(\delta_{\alpha i})(-\delta_{\beta j}\eta_\beta^* V_2) - \vartheta_{ij}(\delta_{\alpha i})(-\delta_{\beta j}\eta_\beta V_2)
\end{aligned}
$$

$$
\dot{\eta}_\alpha^* = 2i\eta_\alpha^* V_2 + \vartheta_{\alpha\beta}\eta_\beta V_2. \tag{4.9}
$$

Observemos las ecuaciones obtenidas para cada una de las variables dinámicas. Los términos de corrección obtenidos para las variables bosónicas están afectados ahora por el potencial fermiónico, mientras que las ecuaciones de movimiento para las variables fermiónicas no sufrieron cambio alguno, es decir, no las alteró el hecho de que ahora hubiera términos bosónicos.

Finalmente, sustituyendo todos los resultados anteriores, se llega a la ecuación de la segunda ley de Newton:

$$\begin{aligned} m\ddot{x}_\alpha = \; & \beta_{\alpha j}\dot{x}_j - [\tfrac{\partial V_1}{\partial x_\alpha} + \eta^*_\beta \eta_\beta \tfrac{\partial V_2}{\partial x_\alpha}] \\ & + m\theta_{\alpha j}[\tfrac{\partial^2 V_1}{\partial x_k \partial x_j}\dot{x}_k + (2i\eta^*_\beta V_2 + \vartheta_{\beta\gamma}\eta_\gamma V_2)\eta_\beta \tfrac{\partial V_2}{\partial x_\alpha} \\ & - \eta^*_\beta(\varphi_{\beta\gamma}\eta^*_\gamma V_2 - 2i\eta_\beta V_2)\tfrac{\partial V_2}{\partial x_\alpha} - \eta^*_\beta \eta_\beta \tfrac{\partial^2 V_2}{\partial x_k \partial x_\alpha}\dot{x}_k]. \end{aligned}$$