SYMMETRY DECOMPOSITION OF MULTI-GLUON INDUCED VERTICES WITHIN THE HIGH ENERGY EFFECTIVE ACTION

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Chapter 1

Introduction

In the study of high energy processes for quark-quark scattering we consider the high energy limit, in which the centre of mass Mandelstam variable s is significantly larger that the momentum transfer t. Moreover, if all momentums are large enough, the coupling α_s is, in comparison, sufficiently small that the model invites perturbation theory. After some significant simplifications, one is able to assume that all contributions to the scattering amplitude are provided by Reggeized gluon (See perhaps [Lipatov, 1995] [Hentschinski, 2012]). In this model, the pole prescription to order k vertices involves computing traces of linear combinations of matrix products $t^{a_1} \cdots t^{a_k}$ where the set $\{t^{a_i}\}_{i=i}^{k}$ is a basis of $\rho(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra $\mathfrak{su}(N_c)$ of color charge, and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a faithful representation of $\mathfrak{g}$ in some vector space V*. This particular computation is done significantly easier for linear combinations that can be rewritten in terms of structure constants and symmetric tensors, for the symmetric tensor vanish to preserv the Bose symmetry, see for example [Hentschinski, 2012]. Our aim is to provide an algorithm to find a basis of the space of all such linear combination satisfying these symmetric properties.

The fact that the factors of the products $t^{a_1} \cdots t^{a_k}$ are a basis of $\rho(\mathfrak{g})$ immediately implies that whenever $i \neq j$ with $i, j \in 1, \dots k$, then $t^{a_i} \neq t^{a_j}$. Thus we may identify a as a permutation on $(1, \dots, k)$. Moreover, we may fix the basis $\{t^{a_i}\}$ of $\mathfrak{g}$ with a preferred ordering $(1, \dots k)$ and consider it to be inherent to the structure of $\mathfrak{g}$, so that we could identify a permutation $a \in S_k$ with the product $t^{a(1)} \cdots t^{a(k)} \in \mathfrak{gl}(V) = \mathrm{End}(V)$.

Although the identification of S_k with this space is nice, it does not 'remember the Lie algebra structure' in terms of the the Lie bracket $[\cdot, \cdot] : \rho(\mathfrak{g}) \times \rho(\mathfrak{g}) \to \rho(\mathfrak{g})$, which in this context is given by the Lie bracket of $\mathfrak{gl}(V)$, i.e. the commutator $[f, g] = f \circ g - g \circ f = fg - gf$. Then we need to remember each factor of the product, but still work with permutation action in the factors, we do this by introducing words, then generate the 'free' algebra on the words with a Lie bracket analog, such objects are known as the Lie polynomials. We use this formulation to workout the desired basis.

*We can consider the fundamental or the adjoint representation

Chapter 2

Fundamentals of Symmetries and QFT

2.1 Groups and symmetries

Suppose $\mathfrak{U}$ is category. Let X be an object of $\mathfrak{U}$, the set of automorphisms of X is the collection of morphisms $f : X \to X$ for which there exist another morphism $f^{-1} : X \to X$ satisfying $f \circ f^{-1} = \mathrm{Id}$, where Id is the identity of X; we denote this set by $\mathrm{Aut}(X)$. We can naturally endow $\mathrm{Aut}(X)$ with a group structure by means of the 'composition' operation

$$\circ : \mathrm{Aut}(X) \times \mathrm{Aut}(X) \to \mathrm{Aut}(X) \text{ given by } (f, g) \mapsto f \circ g.$$

Since automorphisms of X leave the structure of X (viewed as an object of $\mathfrak{U}$) invariant, we may think of $\mathrm{Aut}(X)$ as the group of symmetries of X. More over, let $(G, \cdot)$ be any group. A left action of G on X is a map $\sigma : G \times X \to X$ satisfying that for any $g, h \in G$ and $x \in X$ we have

$$\sigma(g \cdot h, x) = \sigma(g, \sigma(h, x)). \tag{$*$}$$

To express this in a more confusing way, we may say that a left action of G on X induces a group homomorphism of G into $\mathrm{Aut}(X)$. That is, that we can think of the map $\rho : G \to \mathrm{Aut}(X)$ given by $g \mapsto \phi_g$, where $\phi_g : X \to X$ is defined as $\phi_g(x) = \sigma(g, x)$. The condition $(*)$ can now be restated as to require that for any $g, h \in G$ we have $\phi_g \circ \phi_h = \phi_{g \cdot h}$. Sometimes we omit σ and ϕ whenever there is no risk of ambiguity, thus we write gx to refer to $\sigma(g, x) = \phi_g(x)$. . We reach the well-known conclusion that the study of group theory boils down to the study of symmetries.

If G is any group and V is a vector space, the space $\mathrm{Aut}(V)$ is juts the General Linear group $\mathrm{GL}(V)$ consisting of linear invertible transformations of V onto itself. An action σ of G on V induces the map $\rho : G \to \mathrm{Aut}(V) = \mathrm{GL}(V)$, this map is called a representation of G; in this context we refer to V as the representation space of G. The choice of a name should already indicate that we may think of G as identified with a subgroup of $\mathrm{GL}(V)$, a group of matrices. We say that the representation is faithfull, if ρ in injective; which in turn is equivalent to require that ρ send g to a matrix that is not the identity matrix whenever g does not equal the identity element e of G.

2.2 Permutation groups

We give a brief discussion on the Symmetric groups and Alternating groups. A permutation of a finite set of objects $\mathcal{O} = \{o_1 \ldots, o_n\}$ is a bijection σ from $\mathcal{O}$ onto itself. It is on the simplest way of thinking about symmetries, in this case coincides with the notion of an automorphism in the category **Set** of sets. In this category, any two finite sets A and B are isomorphic if and only if they have the same number of elements. Thus it suffices to consider the cases where $\mathcal{O}_n = \{1, \ldots, n\}$.

The set of bijections of $\mathcal{O}_n$ will be denoted S_n. It is easy to see that S_n has exactly $n!$ elements. Note that if $n = 0$, the set of objects $\mathcal{O}_0$ is empty; there is however exactly one bijection on $\varnothing$, the empty map. This agrees with the fact that $0! = 1$. Let G be any finite group, consider the map $f : G \to \mathrm{Aut}(G)$ (Automorphisms in the category **Set**) given by $g \mapsto m_g$, where m_g is the left multiplication map $m_g(h) = g \cdot h$. The fact that G is finite implies that G isomorphic (as a set) to $\mathcal{O}_n$ for some n. Thus we can consider the map $\phi : G \mapsto S_n$ given by $\phi = \psi \circ f \circ \psi^{-1}$, where ψ is the set isomorphism $G \simeq \mathcal{O}_n$.

Theorem 2.2.1 (Cayley's Theorem)**.** *Every finite group is a subgroup of S_n for some $n \in \mathbb{N}$.*

It is thus of great importance to study the permutation groups. We will write explicitly the permutation action in the usual notation, e.g. the permutation $\sigma \in S_n$ given by $(1, \ldots, n) \mapsto (\sigma(1), \ldots, \sigma(n))$ will be denoted as

$$\sigma = \begin{pmatrix} 1 & \cdots & n \\ \sigma(1) & \cdots & \sigma(n) \end{pmatrix}.$$

We will sometimes use cyclic notation, based in the fact that every permutation is a finite product of cyclic permutations, even though this cyclic factorisation is not unique, the fact whether the number of non-trivial cycles in which a permutation decomposes is even or odd is an invariant to any cyclic factorisation. Thus we make a distinction between even and odd permutations. The subgroup of even permutation of S_n is called the alternating groups, and it is denoted by A_n.

The regular representation of a permutation group S_n is given by the free group vector space $\mathbb{C}S_n$ which is the $\mathbb{C}$ vector space spanned by the vectors $\{e_g\}_{g \in S_n}$. The action $\rho : S_n \to \mathrm{GL}(\mathbb{C}S_n)$ is naturally given by $\rho(g)(e_h) = e_{g \cdot h}$ This vector space is $n!$-dimensional. Note, however that is not irreducible, for the 1 dimensional subspace containing the vector $v = \sum_{g \in S_n} e_g$ is invariant under ρ.

Another natural representation is given on the vector space $\mathbb{C}^n$, where S_n acts permuting the basis $\{e_1, \ldots e_n\}$; i.e. $\sigma \cdot e_i = e_{\sigma(i)}$; again the line spanned by $v = \sum_{i=1}^{3} e_i$ is invariant under this action; its complementary subspace $V = \{(z_1, \ldots, z_n) : z_1 + \cdots + z_n = 0\}$ is irreducible; it is called the standard representation of S_n.

Finally, the alternating representation of S_n acts on any vector space by the multiplication by the sign of the permutation; that is $\sigma \cdot v = \mathrm{sgn}(\sigma) v$; where $\mathrm{sgn}(\sigma)$ is 1 for even permutations and -1 for odd permutations.

2.3 Lie groups

Consider the real n-dimensional vector space $\mathbb{R}^n$, the group of symmetries (automorpshims) is $\mathrm{GL}(\mathbb{R}^n) = \mathrm{GL}(n, \mathbb{R})$. We may think of it as a space of matrices considering the usual basis $e^1, \dots e^n$. Thus we have an injective map $\psi : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^{n^2}$. Hence, we can identify $\mathrm{GL}(n, \mathbb{R})$ with the image of ψ. Recall that $\det : \mathbb{R}^{n^2} \to \mathbb{R}$ is a continuous function. Hence, $\mathrm{GL}(n, \mathbb{R})$ is an open subspace of $\mathbb{R}^{n^2}$ and thus a differentiable* manifold.

As we mentioned in the previous section, $\mathrm{GL}(n, \mathbb{R})$ is a group with product rule given by composition, or in this case, matrix multiplication. This operation, viewed as a map $\cdot : \mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ is clearly differentiable. The inversion operation can also be thought as a map $\iota : \mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$; in this context it is given by

$$\iota(A) = A^{-1} = \frac{1}{\det A} \mathrm{adj}(A),$$

where $\mathrm{adj}(A)$ is the adjugate† matrix of A. It is easy to see that ι is also differentiable. In this sense, we say that the group structure of $\mathrm{GL}(n, \mathbb{R})$ is compatible with its structure as a differentiable manifold. This motivates the following definition.

Definition 2.3.1. A **Lie group** is a differentiable manifold G that is also a group, such that the map $G \times G \ni (g, h) \mapsto g \cdot h^{-1}$ is differentiable. A map between Lie groups (or a Lie group homomorphism) is a differentiable group homomorhpism between two Lie groups.

We now describe some typical Lie groups, defined as subsets of $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$. First, we define $\mathrm{SL}(n, R)$ and $\mathrm{SL}(n, \mathbb{C})$ as the invertible real and complex $n \times n$ matrices with unit determinant. Given a bilinear symmetric form g on $\mathbb{R}^n$, we define $\mathrm{O}(g, n, \mathbb{R})$ as the invertible $n \times n$ matrices A preserving g, that is, $g(Av, Aw) = g(v, w)$. When g is the usual inner product, this condition boils down to requiring that $A^T \cdot A = \mathrm{Id}$; we denote it simply by $\mathrm{O}(n)$. Note that if $A \in \mathrm{O}(n)$, then $\det A$ must be either 1 or -1. Since det is a continuous map, $\mathrm{O}(n)$ must have two components, the component containing the identity $\mathrm{O}(n) \cap \mathrm{SL}(n, \mathbb{R})$ is denoted as $\mathrm{SO}(n)$.

Given a bilinear Hermitian form H on $\mathbb{C}^n$, we define $U(H, n, \mathbb{C})$ as the invertible $n \times n$ complex matrices A that preserve H, that is, $H(v, w) = H(Av, Aw)$. When H is the usual complex-inner product, this condition boils down to requiring that $A^\dagger A = \mathrm{Id}$; we denote this group simply by $\mathrm{U}(n)$. We define, analogously to the real case, the group $\mathrm{SU}(n)$ as the component of $\mathrm{U}(n)$ containing the identity, or equivalently $\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C})$. We do not prove that all this are Real‡ Lie groups, but the proofs can be found in [Zee, 2016]. Note also that complex conjugation is not a differentiable complex function, thus $U(n)$ and $SU(n)$ are not complex Lie groups.

We next develop some of the important theorem concerning Lie groups, which as we shall see, motivate the notion of Lie algebras. First we will see that Lie groups (actually any topological group) 'looks' the same everywhere. That is, for any pair of points $p, q \in X$ there is a diffeomorphism $f : X \to X$ that maps x to y and vice-versa; a smooth manifold with this property is said to be **homogeneous**.

Proposition 2.3.2. *Every Lie group is homogeneous.*

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*Here and throughout this work, differentiable means smooth.

†The transpose of the cofactors matrix.

‡They are differentiable manifolds in the real sense and not necessarily in the complex sense

Proof. Let x and y be two group elements, the map $g \mapsto x \cdot g^{-1} \cdot y$ is diffeomorphism, being the composition of diffeomorphisms (inversion and left/right multiplication). This map sends $x \mapsto y$ and $y \mapsto x$. □

This suggests that we could pick any point $g \in G$ and study the group 'from there', in the sense that whenever G satisfies some property with respect to any point $h \in G$, then it also satisfy so in g. So that the study of Lie groups could be done locally. We must agree to fix a point in G, the natural choice is, of course, the identity $e \in G$. To see that, indeed we could study G locally, we prove the following

Proposition 2.3.3. *Let G be a connected Lie group. Then any open neighbourhood of the identity generates G.*

Proof. Let U be an open neighborhood of e. We may assume that $U = U^{-1}$, Let S be the set of finite products of elements of U. We prove that S is clopen, this will suffice. We see immediately that S is open for $gU \subset S$ for any $g \in S$. To show that G is closed we fix $g \notin S$, then gU cannot intersect S for if it did, then we would have $gs = t$ for some $s, t \in S$, which in turn implies that $g \in S$. Hence, S is closed. □

Since G is not only a group, but also a differentiable manifold, we can consider how this proposition extends to the vector fields in G. For any $g \in G$, let m_g be the map of multiplication by g on the left, that is $m_g(h) = g \cdot h$.

Definition 2.3.4. A vector field X of a Lie group G is said to be a **left invariant vector field** if $(m_g)_{*,h} X_h = X_{g \cdot h}$.
Here $(m_g)_{*,h}$ stands for the pushforward of m_g at h, and $X_h = X(h)$ is the tangent vector at h obtained by the evaluation of X at h.

From Proposition 2.3.2 it follows that the tangent spaces $T_g G$ at g and $T_h G$ at h are isomorphic as vector spaces, by means of $(m_{hg^{-1}})_{*,g}$. The requirement of a vector field to be left invariant means that it must be compatible with such isomorphism. In particular, the following diagram commutes

$$\begin{tikzcd} T_e G \arrow[r, "(m_g)_{*,e}"] & T_g G \\ G \arrow[u, "X"] \arrow[r, "m_g"] & G. \arrow[u, "X"'] \end{tikzcd}$$

Now suppose that X is a tangent vector at the identity. We may construct a vector field $\tilde{X}$ in G given by $\tilde{X}_g = (m_g)_{*,e} X$. This vector space is by construction left invariant. More over, it is straightforward to note that vector space of all left invariant vector fields L^G is isomorphic to the tangent space $T_e G$ at the identity.

For any smooth manifold M, let $\Gamma(TM)$ be the space of all smooth vector field in M. We can define the bilinear map $[\cdot,\cdot] : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ given by the commutator $[X, Y]f = X(Yf) - Y(Xf)$ for any smooth function $f : M \to \mathbb{R}$.

Lemma 2.3.5. *Let $f : M \to N$ be a diffeomorphism; let X and Y be C^∞ vector fields in M. Then $f_*[X, Y] = [f_*X, f_*Y]$.*

Proof. Compute. Recall that when f is a diffeomorphism, then f_*X is a well defined vector field, given by $f_*X(q) = X_p \circ f^*$, for any $q \in N$; where $f(p) = q$. □

In particular, this means that whenever X and Y are left-invariant vector fields in a Lie group, then $[X, Y]$ is again left invariant. It is not true, however, that $X \circ Y$ is in general left invariant. If $f : G \to H$ is Lie group map, then it maps the identity e in G to the identity e' in H. Thus $f_{*,e}$ is a linear map from T_eG into $T_{e'}H$; or equivalently a linear map between the vector space of left invariant vector fields in G and H respectively. Indeed let X be a tangent vector at $e \in G$ and $\tilde{X}$ its corresponding left invariant vector field. We define the left invariant vector field $f_*\tilde{X}$ to be the left invariant extension of $f_{*,e}X$. Clearly we have that $[f_*\tilde{X}, f_*\tilde{Y}] = f_*[\tilde{X}, \tilde{Y}]$. Explicitly, we have that f_* preserves the commutator, considered as a linear map between the spaces of left invariant vector fields in G and H respectively. Furthermore we have the following very important results left without proof.[§]

Theorem 2.3.6. *Let G and H be Lie groups with G connected and simply connected. A linear map $f : T_eG \to T_{e'}H$ from the tangent space at the identity of G into the tangent space at the identity at H is the differential at the identity of a unique Lie group map if and only if f preserves the commutator.*

2.4 Lie algebras

In view of the fact that the tangent space T_eG takes such an important role in the local description of a Lie group, we study this in detail. Since T_eG is isomorphic as a vector space with the left invariant vector fields in G, under this isomorphism we may pullback the bilinear map $[\cdot, \cdot] : L^G \times L^G \to L^G$ to the map $\cdot, \cdot : T_eG \times T_eG \to T_eG$. Fortunately we can express this map explicitly. Let $X, Y \in T_eG$ and $\tilde{X}, \tilde{Y}$ be their left invariant vector field extensions. Then X, Y is given by the evaluation of $[\tilde{X}, \tilde{Y}]$ at e; as one should expect. The structure $\mathfrak{g} = (T_eG, \cdot, \cdot)$ is called the Lie algebra of G. More over, we could define the notion of Lie algebras without a underlying Lie group.

Definition 2.4.1. A **Lie algebra** is a vector space $\mathfrak{g}$ with a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying $[X, Y] = -[Y, X]$ and the so-called "Jacobi identity"

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

for all $X, Y, Z \in \mathfrak{g}$. The map $[\cdot, \cdot]$ is usually called the Lie bracket.

A vector subspace $\mathfrak{h} \subset \mathfrak{g}$ is said to be a **left ideal** of $\mathfrak{g}$ if for every $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$ their Lie bracket $[X, Y]$ lies in $\mathfrak{h}$. We may write this as $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$. We may analogously define right and two sided ideals. A lie algebra whose only ideals are trivial is called **simple**. In

[§]For a proof, see for example [Fulton and Harris, 2004] or [Zee, 2016] other words, we say that $\mathfrak{g}$ is simple if whenever $\mathfrak{h}$ is an ideal, then either $\mathfrak{h} = \mathbf{0}$ or $\mathfrak{h} = \mathfrak{g}$. Finally, we say that a Lie algebra $\mathfrak{g}$ is **semi-simple** if it is the Lie algebra direct sum

$$\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \cdots$$

of simple Lie algebras. The Lie algebra direct sum is defined in the natural way. Explicitly, if $(\mathfrak{h}_1, [\cdot,\cdot]_1)$ and $(\mathfrak{h}_2, [\cdot,\cdot]_2)$ are Lie algebras, then $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ is the vector space direct sum together with the map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ given by

$$[X, Y] = \begin{cases} [X, Y]_i, & \text{if } i = j \\ 0, & \text{if } i \neq j; \end{cases}$$

where $X \in \mathfrak{h}_i$ and $Y \in \mathfrak{h}_j$. Clearly this suffices to describe the Lie bracket of $\mathfrak{g}$.

If G is a subgroup of $\mathrm{GL}(n, \mathbb{C})$ or $\mathrm{GL}(n, \mathbb{R})$ we may identify its Lie algebra with a linear subspace of $\mathrm{End}(n, \mathbb{C})$ or $\mathrm{End}(n, \mathbb{R})$ respectively. Then the composition XY of two elements $X, Y \in \mathfrak{g}$ is defined, however, it need not be an element of $\mathfrak{g}$. The commutator, on the other hand, is always again an element of $\mathfrak{g}$. Indeed the Lie bracket in this context is given by the commutator $[X, Y] = XY - YX$. As a matter of fact, every Lie algebra is the Lie algebra of a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$ for some n (For a proof see [Bourbaki, 1975]).

2.5 The exponential map

Let $X \in \mathfrak{g}$ be a tangent vector at the identity e of some Lie group G; let $\tilde{X}$ be its left invariant vector field extension. It is a fundamental fact of differential geometry that there is an integral curve γ from some interval about 0 into G such that $\gamma(0) = e$ and $\gamma'(t) = \tilde{X}_{\gamma(t)}$. More over, for any other path $\alpha : J \to G$ satisfying $\alpha(0) = e$ and $\alpha'(t) = \tilde{X}_{\alpha(t)}$, then the paths γ and α coincide in $I \cap J$.

Proposition 2.5.1. *Let $\gamma : I \to G$ be an integral curve of the left invariant extension $\tilde{X}$ of some tangent vector at the identity. Then γ extends uniquely to all of $\mathbb{R}$ in such a way that it is a Lie group homomorphism.*

Proof. Let us fix some $t > 0$ small enough and J be an appropriate interval about 0 so that the maps $\phi_t(s) = \gamma(s + t)$ and $\psi_t(s) = \gamma(s) \cdot \gamma(t)$ are well defined. Clearly $\phi_t(0) = \psi_t(0)$ and by the left invariance of $\tilde{X}$ their derivatives at 0 also coincide. Thus by the uniqueness of the integral curves we have $\psi_t = \phi_t$ for small enough t. Then γ extends uniquely to $\mathbb{R}$ satisfying $\gamma(s + t) = \gamma(s) \cdot \gamma(t)$. $\square$

Thus for every tangent vector X at the identity there exists an associated subgroup $H \subseteq G$ given by the image of the Lie group homomorphism $\gamma_X : \mathbb{R} \to G$ that has X whose derivative at 0 coincides with X. Such subgroup is called the one parameter subgroup of G with tangent vector X at the identity. Clearly if $Y = \lambda X$ for some scalar λ; then their one parameter subgroups coincide; indeed by the uniqueness of integral curves we have

$$\gamma_Y(s) = \gamma_{\lambda X}(s) = \gamma_X(\lambda s).$$

We define the **exponential map** as

$$\exp : \mathfrak{g} = T_eG \to G, \qquad\qquad X \mapsto \gamma_X(1).$$

It is not hard to show that its differential $(\exp)_{*,0} : \mathfrak{g} \to \mathfrak{g}$ is an isomorphism. Indeed it is the identity. This implies, in particular, that for any Lie group homomorphism $f : G \to H$ the following diagram commutes

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "f_*"] \arrow[d, "\exp"'] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "f"] & H. \end{tikzcd}$$

By the inverse function theorem, exp restricts to a diffeomorphism that maps a neighbourhood of 0 onto a neighbourhood of $e \in G$. Since every neighbourhood of the identity generates the connected component of the identity. Then $\mathfrak{g}$ is a linear space inherent to the structure of G that generates G (via the exponential map). In fact, $\mathfrak{g}$ remembers much more about the group structure.

Theorem 2.5.2. *The exponential map satisfies the following properties for all $X, Y \in \mathfrak{g}$ and $\alpha, \beta \in \mathbb{R}$(or $\mathbb{C}$).*

1. $\exp(tX) = \gamma_X(t)$;
2. $\exp((s+t)X) = \exp(sX) \cdot \exp(tX)$;
3. $\exp(-X) = \exp(X)^{-1}$;
4. $\exp(X) \cdot \exp(Y) = \exp(X)\exp(Y)$ *if* $[X, Y] = 0$.

By the invertibility of exp about 0; then for $g \in G$ sufficiently near e; there exists $X \in \mathfrak{g}$ such that $g = \exp(X)$. The map $\log : G \to \mathfrak{g}$ is the local inverse to exp, indeed it is given by $g \mapsto X$. We have the next very important theorem without proof.

Theorem 2.5.3. *Baker-Hausdorff-Campbell Let $X, Y \in \mathfrak{g}$. Then there exists some $Z\mathfrak{g}$ such that $\exp(X) \cdot \exp(Y) = \exp(Z)$ and it is given by*

$$Z = X + Y + \frac{1}{2}[X, Y] - \frac{1}{12}[[X, Y], X] + \frac{1}{12}[[X, Y], Y].$$

As a final remark. Since every Lie algebra can be embedded into End(V) for some vector space V. Then it is worth to look at $\exp : End(V) \to GL(V)$. We easily obtain the final result of this section.

Theorem 2.5.4. *For a matrix Lie algebra End(V) with $[X, Y] = XY - YX$ with $GL(V)$ as its corresponding Lie group, the exponential map and its inverse are given by*

$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}, \qquad \log(g) = \sum_{n=0}^{\infty} (-1)^n \frac{(g - Id)^{n+1}}{(n+1)!}.$$

2.6 Lagrangian density in Quantum Field Theory

We consider now the formulation of Physical theories taking into consideration the Quantum and Relativity theories. First, we consider a 4-dimensional locally Minkowski smooth manifold $\mathcal{M}$, that is, there is a Lorentzian metric defined on $\mathcal{M}$. That is, around any point q of $\mathcal{M}$, we can find a coordinate system $x = (x_0, x_1, x_2, x_3)$ such that there is pseudo-riemannian metric g such that in this coordinate basis, it takes the matrix form

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We are, of course, assuming natural units $\hbar = c = 1$. Now, for every particle there is an associated tensor field ϕ in $\mathcal{M}$; this means that for every point $q \in \mathcal{M}$, $\phi(q)$ is a multilinear real (or complex) valued transformation acting on

$$\underbrace{V \otimes \cdots V}_{k-\text{times}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l-\text{times}},$$

where V is the tangent space $T_q\mathcal{M}$. Such field is said to be $k-$covariant and $l-$contravariant, or (k, l)-tensor. We will, however write equations and define objects in the component notation. That is, we will write $\phi_{\mu_1 \cdots \mu_k}{}^{\nu_1, \cdots \nu_l}$ to the corresponding component in the coordinates x such that g is as above. Explicitly we have

$$\phi_{\mu_1 \cdots \mu_k}{}^{\nu_1, \cdots \nu_l} = \phi(x_{\mu_1}, \ldots, x_{\mu_k}, x^{\nu_1}, \ldots, x^{\nu_l}),$$

where x^ν is the dual of x_ν, that is, $x^\nu(x_\mu) = \delta^\nu_\mu$. The metric g defines an identification of covariant and contravariant tensors. If A is a covariant tensor with one index A_μ, we could define a contravariant tensor B as

$$B^\nu = g^{\mu\nu} A_\mu,$$

where g with upper indices denote the components of the inverse of the matrix g[¶]. Since we write tensors only with their indices, we keep the letter A for the tensor B; the placement of the indices gives the context of how to interpret the tensor.

We now think in a general setting that the fields are maps $\phi : \mathcal{M} \to \Sigma$, where Σ is the corresponding tensor bundle. A Lagrangian density is a function $\mathcal{L} : T\Sigma \times \mathcal{M} \to \mathbb{R}($ or $\mathbb{C})$. The action $S : \mathcal{C} \to \mathbb{C}$, where $\mathcal{C}$ is the infinite dimensional manifold of all fields $\phi : \mathcal{M} \to \Sigma$ is given by

$$S[\phi] = \int_{\mathcal{M}} \mathrm{d}^4\mathbf{x}\; \mathcal{L}(\phi(x), d\phi(x), x).$$

The least action principle states that the fields corresponding to the physical system are the fields that minimise the action S. To derive the Euler-Lagrange equation it suffices to consider the fields in a coordinate chart.

[¶] In this formulation the inverse of g coincides with g.

Let $\phi \in \mathcal{C}$, a variation of S at ϕ is the derivative $d_\phi S \in T_\phi \mathcal{C}$, it is usual to find this object writen as $\frac{\delta S}{\delta \phi}$. Let $X \in T_{\phi(x)}\Sigma$; a point-wise variation[‖] of S at x is given by

$$\frac{\delta S}{\delta \phi(x)} X = \frac{\delta S}{\delta \phi}(\delta_x \cdot X),$$

where δ_x is the delta function centered at $x \in \mathcal{M}$. Finding fields ϕ for which variations of S vanish is analogue to the Euler Equations for the Lagrangian mechanics, indeed we have

Theorem 2.6.1. *The fields $\phi \in \mathcal{C}$ minimise the action if and only if they satisfy en Euler Lagrange equations*

$$\frac{\delta \mathcal{L}}{\delta \phi} = \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \right).$$

Proof. The computation relies in a series of differentiation under the integral and integration by parts. We evualte $d_\phi S$ at some tangent vector $X \in T_\phi \mathcal{C} = \mathcal{C}$

$$\begin{aligned} d_\phi S(X) &= \int_{\mathcal{M}} \mathrm{d}^4 x \; d_\phi \mathcal{L}(X) = \int_{\mathcal{M}} \mathrm{d}^4 x \frac{\delta \mathcal{L}}{\delta \phi}(X) + \frac{\delta \mathcal{L}}{\delta(d\phi)}(dX) \\ &= \int_{\mathcal{M}} \mathrm{d}^4 x \left(\frac{\delta \mathcal{L}}{\delta \phi} - d \frac{\delta \mathcal{L}}{\delta(d\phi)} \right)(X) && \text{After integration by parts} \end{aligned}$$

The variation of S at ϕ written in coordinate yields, written in coordinates yield the Euler Lagrange equations. □

[‖] Physicists call this a functional derivative.

Chapter 3

Pole prescription in multi-gluon vertices

3.1 The Lipatov effective action

In the perturbative expansion of QCD in the high energy limit, it usual to turn to the high energy factorization and the BFKL evolution. In this model, there is a significant simplification given by the large centre of mass energy $\sqrt{s}$. The series expansion, taking into account the leading logarithmic terms and nearly logarithmic terms are described by a t-channel degree of freedom: the reggeized gluon. To study phenomenology in the high energy limit of QCD, we consider the Lipatov's effective action.

We consider a parton-parton elastic scattering process $p_a + p_b \to p_1 \to p_2$ such that p_a and p_b are light-like momenta; i.e. $p_a^2 = p_b^2 = 0$. We define the light-like four-vectors $n^{\pm}$ with $n^+ \cdot n^- = 2$ given by the rescaling of the partons momenta by a factor of $2/\sqrt{s}$. That is, $n^+ = 2p_b/\sqrt{s}$ and $n^- = 2p_a/\sqrt{s}$. The Lipatov's effective action is given by $S_{\text{eff}} = S_{\text{QCD}} + S_{\text{ind}}$, where S_{QCD} is the QCD action and S_{ind} is an induced term that describes a coupling of a reggeized gluon field $A_{\pm} = -it^a A_{\pm}^a$ to the gluonic field $\nu^{\mu} = -it^a \nu_{\mu}^a$; where a runs over an index set J such that $\{t^a\}_J$ is a basis for a faithful representation of $\mathfrak{su}(N)$. Explicitly, it is given by

$$S_{\text{ind}}[\nu, A] = \int_{\mathcal{M}} \operatorname{tr}\Big[(W_+(\nu) - A_+)\partial_{\perp}^2 A_-\Big] + \int_{\mathcal{M}} \operatorname{tr}\Big[(W_-(\nu) - A_-)\partial_{\perp}^2 A_+\Big].$$

The functionals $W_{\pm}$ take into account the infinite couplings of gluon fields with the reggeized gluon field, these are defined as

$$W_{\pm}[\nu] = \nu_{\pm} \frac{1}{D_{\pm}} \partial_{\pm}, \qquad \text{where} \qquad D_{\pm} = \partial_{\pm} + g\nu_{\pm}.$$

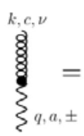
The $\pm, \perp$ decomposition of a general four-vector k is given by the Sudakov decomposition $k = k^+ n/2 + +k^- n^-/2 + k_{\perp}$. For actual calculations, it is worthwhile to look at the perturbative expansion of the $W_\pm$, it reads

$$W_\pm[\nu] = \sum_{m=0}^{\infty}(-1)^m g^m \nu_\pm\big(\frac{1}{\partial_\pm}\nu_\pm\big)^m.$$

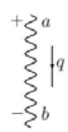
In the original formulation of the Lipatov effective action, there is no implied choice for the pole prescription for the regularization of the terms $1/\partial_\pm$ at zero. At high energies, the ordering of longitudinal amplitudes imply the constrain

$$\partial_+ A_- = \partial_- A_+ = 0.$$

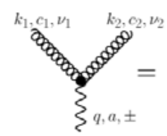
The Feynman rules for this effective action have been determined in [?]. These are given by the usual QCD Feynman rules and infinitely many such for the coupling of a reggeized gluon and a some number of gluonic fields. We depict them using a wavy 'photon-like´ line for the reggeized gluon.



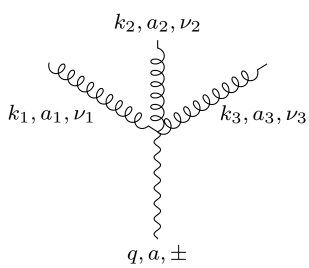
$$= -i\boldsymbol{q}^2\delta^{ac}(n^\pm)^\nu,$$



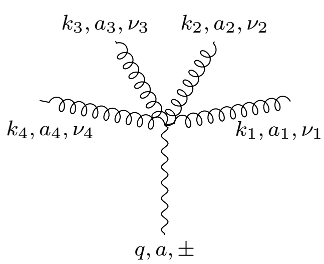
$$= \delta^{ab}\frac{i/2}{\boldsymbol{q}^2}$$



$$= g f^{c_1 c_2 a}\frac{\boldsymbol{q}^2}{k_1^\pm}(n^\pm)^{\nu_1}(n^\pm)^{\nu_2},$$



$$= ig^2\mathbf{q}^2\Big(\frac{f^{a_3a_2e}f^{a_1ea}}{k_3^\pm k_1^\pm} + \frac{f^{a_3a_1e}f^{a_2ea}}{k_3^\pm k_2^\pm}\Big)(n^\pm)^{\nu_1}(n^\pm)^{\nu_2}(n^\pm)^{\nu_3},$$



$$= g^3\mathbf{q}^2\Big[\frac{f^{a_4a_3e_2}}{k_4^\pm}\Big(\frac{f^{e_2a_1e_1}f^{a_2e_1a}}{k_2^\pm(k_1^\pm+k_2^\pm)} + \frac{f^{e_2a_2e_1}f^{a_1e_1a}}{k_1^\pm(k_1^\pm+k_2^\pm)}\Big)$$

$$+ \frac{f^{a_4a_1e_2}}{k_4^\pm}\Big(\frac{f^{e_2a_2e_1}f^{a_3e_1a}}{k_3^\pm(k_3^\pm+k_2^\pm)} + \frac{f^{e_2a_3e_1}f^{a_2e_1a}}{k_2^\pm(k_3^\pm+k_2^\pm)}\Big)$$

$$+ \frac{f^{a_4a_2e_2}}{k_4^\pm}\Big(\frac{f^{e_2a_1e_1}f^{a_3e_1a}}{k_3^\pm(k_1^\pm+k_3^\pm)} + \frac{f^{e_2a_3e_1}f^{a_1e_1a}}{k_1^\pm(k_1^\pm+k_3^\pm)}\Big)\Big](n^\pm)^{\nu-1}(n^\pm)^{\nu_2}(n^\pm)^{\nu_3}.$$

Here we also assume that the sum over gluonic momenta vanishes in the $\pm$ direction; i.e. $k_1^\pm = 0$ for the g induced vertex, $k_1^\pm + k_2^\pm = 0$ for the g^2 induced vertex and similarly for the higher order vertices. This vertices maintain Bose symmetry. The f's are the structure constants of the Lie algebra $\mathfrak{su}(N)$ defined in the physics convention as $[t^a, t^b] = \sum_{c\in J} if^{abc}t^c$, of course, we may omit the summation symbol.

3.2 Pole prescription to the first order vertices

It is usual to interpret the regularization of the induced g vertices as Cauchy principal values, so that we may regulate the factor $\frac{1}{k^\pm}$ by replacing it with the factor

$$\frac{1}{[k^\pm]} = \frac{1}{2}\left(\frac{1}{k^\pm + i\epsilon} + \frac{1}{k^\pm - i\epsilon}\right).$$

This choice has the advantage that it maintains the symmetry of the vertex without pole prescription; in particular both Bose-symmetry of the unregulated vertex and its anti-symmetry under the substitution $k^\pm \to -k^\pm$ are kept. The latter property is of importance as it can be directly related to negative signature of the reggeized gluon, see [**?**]. A straightforward extension of the Cauchy principal value prescription to higher order vertices interprets every separate pole as a Cauchy principal value. However, at least if naively applied to the vertices in Figs. 2 and 3, such a prescription violates Bose-symmetry and can lead to wrong results, see for instance [Lipatov, 1995]. This is mainly due to the fact that Cauchy principal values do not obey the eikonal identity in an algebraic sense. One has instead an additional term containing the product of two delta-functions,

$$\frac{1}{[k_1^\pm][k_1^\pm + k_2^\pm]} + \frac{1}{[k_2^\pm][k_1^\pm + k_2^\pm]} = \frac{1}{[k_1^\pm][k_2^\pm]} + \pi^2 \delta(k_1^\pm)\delta(k_2^\pm)$$

Chapter 4

The antysymmetric sector

4.1 A generalisation of the Jacobi identity

We make a short excursion into some abstract formulation of the Jacobi identity. We begin by rewriting the Jacobi identity as

$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]. \tag{$*$}$$

With this form of the Jacobi identity, we may view $Ad(X) = [X, \cdot]$ as acting on the vectors Y, Z and $[Y,Z]$. Additionally we may think of $[\cdot,\cdot]$ as a bilinear product in $\mathfrak{g}$. With this image, the equation $(*)$ implies that Ad satisfies the product rule (Leibniz rule) with respect to the Lie bracket. Indeed we have the following definition.

Definition 4.1.1. Let $(A,\cdot)$ be an algebra*. A linear map $D : A \to A$ is called a derivation in A if it satisfies the Leibniz rule; i.e. for all $a, b \in A$ we have

$$D(a \cdot b) = D(a) \cdot b + a \cdot D(b).$$

Thus we may think of the Jacobi identity as a statement of $Ad(X)$ being a derivation in $\mathfrak{g}$ for all $X \in \mathfrak{g}$. We can make some calculations using this formulation. For example, the next result will come in handy.

Lemma 4.1.2. *Let $(A,\cdot)$ be an algebra, let $X : A \to A$ be a derivation and $a_1, \dots, a_n \in A$. Then $X(a_1 \cdots a_n) = \sum_{i=1}^{n} a_1 \cdots X(a_i) \cdots a_n$.*

Proof. The calculation follows easily from a repeated application of the Leibniz rule. □

In particular this means that, in the case where $A := \mathfrak{g}$ is some Lie algebra and the product $\cdot$ is given by the Lie bracket, we have that

$$[X,[Y_1[Y_2,\cdots[Y_{n-1},Y_n]\cdots]]] = \sum_{i=1}^{n}[Y_1,[Y_2,\cdots[Y_{i-1},[[X,Y_i],[Y_{i+1},\cdots]\cdots]]]$$

*Products are written without parenthesis, but are meant to be grouped from right to left.

4.2 Lie Polynomials

We begin by defining some terms that will be heavily used in this work. Let A be a finite set[†], we will call this set an alphabet (later on we will set $A = \{1, \dots k\}$) and the elements of the alphabet are called letters. A finite sequence $w = (a_1, a_2, \dots, a_m)$ of letters is called a word, we will denote this sequence by juxtaposition, i.e. $w = a_1 a_2 \cdots a_m$, the length $\|w\|$ of a word w is the number of letters in the sequence it defines.

Let $\mathrm{w}(A, k)$ be the set of words of length k and consider the space of all formal linear combinations $\mathbb{C}\mathrm{w}(A, k)$. The elements of this space are called $k-$polynomials on A over $\mathbb{C}$, we will simply call them $k-$polynomials or just polynomials when there is no ambiguity on the choice of k. There is a natural action of $\mathfrak{S}_k$ on $\mathbb{C}\mathrm{w}(A, k)$ given by the permutation of the letters, i.e. for $\sigma \in \mathfrak{S}_k$, it acts in each word $w = a_1 \cdots a_k$ as $\sigma \cdot w = a_{\sigma(1)} \cdots a_{\sigma(k)}$; clearly this suffices to describe the action in the space of polynomials. We set $\mathrm{w(A)} = \bigcup_{k \geq 0} \mathrm{w}(A, k)$ as the set of all words on A and $\mathbb{C}\mathrm{w}(A)$ as the space of all polynomials on A over $\mathbb{C}$. The latter can be endowed with a Lie algebra structure with the Lie bracket given as the commutation of concatenation; that is, for any two words P and Q as $[P, Q] = PQ - QP$.

Now, we will set $A = \{1, \dots, k\}$, the words $w = a_1 \cdots a_m$ on A can now be identified with the matrix products $t^{a_1} \cdots t^{a_m} \in \mathrm{End}(V)$. We extend this identification linearly to the space $\mathbb{C}w(A, k)$. The Lie bracket can be in particular evaluated in the letters, so that the expression

$$a_1 \cdots [a_i, a_{i+1}] \cdots a_k \quad \text{is identified with} \quad t^{a_1} \cdots t^{a_k} - t^{a_1} \cdots t^{a_{i+1}} t^{a_i} \cdots t^{a_k}.$$

It is worth mention a subtlety in this formulation. Note that the expression on the left in the identification above can be expressed in terms of the Lie bracket $[\cdot, \cdot]$ in $\mathfrak{g}$, and thus we could make use of the structure constants to write it a linear combinations of products of $k-1$ matrices; in this sense we could consider $a_1 \cdots [a_i, a_{i+1}] \cdots a_k$ to be an element of $\mathbb{C}\mathrm{w}(A, k-1)$. However we decidedly force the bracket to 'forget' the structure constants and therefore we consider it as an element of $\mathbb{C}\mathrm{w}(a, k)$, we could say that we are dealing with free a *Free Lie ring*. This is done in order to avoid ambiguity on the dimension on some spaces in which the commutator relations may add constraints and thus eliminate degrees of freedom[‡].

We now define Lie words $\mathrm{Lw}(A, k) \subseteq \mathbb{C}\mathrm{w}(A, k)$ inductively on their length. The Lie 1-words coincide with words on A of length 1; i.e. $\mathrm{Lw}(A, 1) = \mathrm{w}(A, 1)$; now suppose the Lie ℓ-words are defined for all $\ell < k$. We set $\mathrm{Lw}(A, k)$ to be the set of all commutators $[P, Q]$ of Lie words P and Q belonging to $\mathrm{Lw}(A, r)$ and $\mathrm{Lw}(A, s)$ respectively that satisfy $r + s = k$, $r \neq k$ and $s \neq k$. The order $\|P\|_L$ of a Lie word P equals k if $P \in \mathbb{C}\mathrm{w}(A, k)$. We will use the following shortcut notation defined inductively on regular words over A as

$$[a] = a \quad \text{for } \|a\| = 1, \qquad [Pa] = [[P], a] \quad \text{for } \|a\| = 1 \text{ and } \|P\| \geq 1.$$

We also set $[P; Q] = [[P], [Q]]$. For the sake of clarity, we show three examples

$$[1234] = [[[1,2],3],4], \quad [12;3,4] = [[1,2],[3,4]], \quad [123;4,5,6] = [[[1,2],3],[[4,5],6]].$$

[†]Some authors allow A to be infinite, but this is beyond the scope of this work

[‡]This could also be avoided by taking the matrix products to be tensor products, and thus no real multiplication of matrices take place.

Whenever there are some subsequent letters that we do not want to bracket, but rather keep them as a word, we will use parenthesis; e.g. $[12(34)] = [[1,2],(34)]$. Note that whenever a bracket has a comma inside it is meant to denote a commutator, in contrast with our shortcut notation. In addition for already bracketed expressions (both shortcut notation and commutator), we assume parenthesis are around the brackets; e.g

$$[12[345][6,(78)]] = [[[1,2],[[3,4],5]],[6,(78)]].$$

Lie k-polynomials are naturally defined as formal linear combinations of Lie words of order k over $\mathbb{C}$. The space of Lie k-polynomials will be denoted $\mathcal{L}^k$. By construction, the union of $\mathcal{L} = \bigcup_{k\geq 0} \mathcal{L}^k$ is the smallest Lie subalgebra of $\mathbb{C}\mathrm{w}(A)$[§]. $\mathcal{L}$ will be called the Jacobi sector of $\mathbb{C}\mathfrak{G}_k$, elements of this space are simply called Lie polynomials.

Proposition 4.2.1 (Baker's identity [Reutenauer, 1993])**.** *Let P and Q be words on A. Them* $[P[Q]] = [[P],[Q]]$

Proof. Let $P = a_1 \cdots a_k$ and $Q = b_1 \cdots b_m$. We proceed by induction on m; clearly the result holds by definition for $m = 1$. Now we suppose that it also does for $m = s$. We write $Q = Rb_{s+1}$, where $R = b_1 \cdots b_s$. Then

$$\begin{aligned}
[P[Q]] &= [P[Rb_{s+1}]] = [P[R]b_{s+1} - Pb_{s+1}[R]] \\
&= [[P,[R]],b_{s+1}] - [[Pb_{s+1}],[R]] \\
&= [[[P],[R]],b_{s+1}] - [[[P],b_{s+1}],[R]] \\
&= [[P][R] - [R][P],b_{s+1}] - [[P]b_{s+1} - b_{s+1}[P],[R]] \\
&= [P][R]b_{s+1} - [R][P]b_{s+1} - b_{s+1}[P][R] + b_{s+1}[R][P] \\
&\qquad - [P]b_{s+1}[R] + [R][P]b_{s+1} + b_{s+1}[P][R] - [R]b_{s+1}[P] \\
&= [P][R]b_{s+1} + b_{s+1}[R][P] - [P]b_{s+1}[R] - [R]b_{s+1}[P] \\
&= [[P],[[R],b_{s+1}]] = [[P],[Q]].
\end{aligned}$$

□

Corollary 1. If P is a Lie word of order k, then $[P] = kP$

Proof. □

4.2.1 Shuffles and generalized Jacobi identities

An (s,t)-Shuffle is pair (γ,η) of strictly increasing functions

$$\gamma : \{1,\dots s\} \to \{1,\dots s+t\} \qquad \text{and} \qquad \eta : \{1,\dots,t\} \to \{1,\dots,s+t\},$$

with disjoint images, we also require that $\gamma(1) = 1$. We may think of (γ,η) as the resulting order of a shuffle of two decks of cards, each of which contains s and t cards respectively. The images of γ and η are the new positions of their cards. The set of all such (s,t)-Shuffles is denoted $\mathrm{Sh}(s,t)$. The next lemma and theorem are stated and proved in [Alekseev and Ivanov, 2016]. We present alternative proofs using our conventions and terminology.

[§]This is the usual definition of Lie polynomials.

Lemma 4.2.2. *For an m-word $P = [a_1 \cdots a_m]$ we have*

$$P = \sum_{i=0}^{m-1} \sum_{(\gamma,\eta)} (-1)^i a_{\eta(i)} \cdots a_{\eta(1)} a_{\gamma(1)} \cdots a_{\gamma(m-i)}{}^{\P}$$

Proof. We proceed by induction. For $m = 2$ the result holds. Now suppose it does for $m = k$. Then consider the word Q with $\|Q\| = k$ such that $P = [Q, a_{k+1}]$. Now using the induction hypothesis we may write P as

$$P = \sum_{i=0}^{k-1} \sum_{(\gamma,\eta)} (-1)^i a_{\eta(i)} \cdots a_{\eta(1)} a_{\gamma(1)} \cdots a_{\gamma(k-i)} a_{k+1} - a_{k+1} a_{\eta(i)} \cdots a_{\eta(1)} a_{\gamma(1)} \cdots a_{\gamma(k-i)}. \tag{$*$}$$

For each $(\gamma', \eta') \in \mathrm{Sh}(k+1-i, i)$ we must have that either $\gamma'(k+1-i) = k+1$ or $\eta'(i) = k+1$.

For the former case, note that we may extend each $(k-i, i)$ shuffle (γ, η) to a a $(k+1-i, i)$ shuffle (γ', η') by taking

$$\gamma'(r) = \begin{cases} k+1, & r = k+1-i; \\ \gamma(r), & r \neq k+1-i; \end{cases} \qquad \eta'(r) = \eta(r).$$

This defines a bijection from $(k-i, i)$ shuffles to the $(k+1)$ onto the $(k+1-i, i)$ shuffles satisfying the case at hand. Thus we may rewrite the first term in $(*)$ as

$$(-1)^i a_{\eta'(i)} \cdots a_{\eta'(1)} a_{\gamma'(1)} \cdots a_{\gamma'(k-i)} a_{\gamma'(k+1-i)},$$

where the sum is taken over all $(k+1-i, i)$ shuffles that satisfy this first case.

For the case $\eta'(i) = k+1$ note that this implies, in particular that $i \geq 1$; let us set $j = i-1$, note that $0 \leq j \leq k-1$. We may extend each $(\gamma, \eta) \in \mathrm{Sh}(k-j, j)$ to a $(k+1-i, i)$ shuffle (γ', η') by defining

$$\gamma'(r) = \gamma(r), \qquad \eta'(r) = \begin{cases} k+1, & r = i; \\ \eta(r) & r \neq i. \end{cases}$$

This defines a bijection from $\mathrm{Sh}(k-j, j)$ onto the $(k+1-i, i)$ shuffles satisfying the latter case. Now we rewrite the second term in $(*)$ as

$$(-1)^{i-1} a_{\eta'(i)} \cdots a_{\eta'(1)} a_{\gamma'(1)} \cdots a_{\gamma'(k-i)} a_{\gamma'(k+1-i)},$$

note that the factor $(-1)^{i-1}$ is taking into account the change of index $i \to j = i-1$. Again, the summ is taken over the $(k-i, i)$ shuffles (γ', η') satisfying $\eta(i) = k+1$. □

Theorem 4.2.3. *Let Q be a k-word and let P be the $n+k$-Lie word $P = [a_1 \cdots a_m(Q)]$ we have*

$$P = \sum_{i=0}^{m-1} \sum_{(\gamma,\eta)} (-1)^{i-1} [(Q) a_{\eta(i)} \cdots a_{\eta(1)} a_{\gamma(1)} \cdots a_{\gamma(m-i)}].$$

¶Whenever (γ, η) is the index of a summation, we assume them to run from $\mathrm{Sh}(s, t)$ for the explicit values of s and t shown in the summation.

Proof. Note that we could write $P = \mathrm{ad}([a_1 \cdots a_m])(Q)$. Since ad is a Lie algebra homomorphism, we have that $\mathrm{ad}([a_1 \cdots a_m]) = [\mathrm{ad}(a_1) \cdots \mathrm{ad}(a_m)]$; where each $\mathrm{ad}(a_i)$ is taken as a single letter. Using the previous lemma we get

$$\begin{aligned} P &= \sum_{i=0} \sum_{(\gamma,\eta)} (-1)^i \Big(\mathrm{ad}(a_{\eta(i)}) \circ \cdots \circ \mathrm{ad}(a_{\eta(1)}) \circ \mathrm{ad}(a_{\gamma(1)}) \circ \cdots \circ \mathrm{ad}(a_{\gamma(m-i)}) \Big)(Q) \qquad (**) \\ &= \sum_{i=0} \sum_{(\gamma,\eta)} (-1)^i [a_{\eta(i)}, [\ldots, [a_{\eta(1)}, [a_{\gamma(1)}, [\ldots, [a_{\gamma(m-i)}, Q] \ldots]. \end{aligned}$$

Now, note that if we flipped all the expression inside the brackets we could turn the 'nested to the right commutator' to the shortcut [ ], but we would also have a factor $(-1)^m$ due to the m flips inside each bracket. Then we have

$$P = \sum_{i=0} \sum_{(\gamma,\eta)} (-1)^{i+m} [(Q) a_{\gamma(m-i)} \cdots a_{\gamma(1)} a_{\eta(1)} \cdots a_{\eta(i)}].$$

Finally we note that we can define a map $\phi : \mathrm{Sh}(m-i, i) \to \mathrm{Sh}(m-j, j)$, where $j = m-1-i$. This map is given by $(\gamma, \eta) \mapsto (\alpha, \beta)$, where

$$\alpha(r) = \begin{cases} 1, & r = 1; \\ \eta(r-1), & 2 \le r \le i+1 = m-j; \end{cases} \qquad \beta(r) = \gamma(r+1), \quad 1 \le r \le m-1-i = j.$$

It easily verified that ϕ is a bijection. Thus we can rewrite P accordingly, taking into account the change of index $i \to j = m-1-i$, we have an extra factor $(-1)^{m-1}$ inside the sum. This completes the proof. $\square$

Let $w = a_1 \cdots a_m$ be a regular word on A of length m; we define the m-polynomial

$$\phi(w) = w + a_j [a_1 \cdots a_{j-1}].$$

If w can be written as the concatenation of two words r and s of lengths j and $m-j$ respectively for $2 \le j \le m$. We define the m-polynomial

$$\theta_j(w) = \phi(r)s$$

Lemma 4.2.4. *Let $w \in w(A, m)$. Then $[\phi(w)] = 0$.*

Proof. By proposition 4.2.1, we have that $[a_m [a_1 \cdots a_{m-1}]] = [a_m, [a_1 \cdots a_{m-1}]]$, which equals

$$\sum_{i=0}^{m-2} \sum_{(\gamma,\eta)} (-1)^i a_m a_{\eta(i)} \cdots a_{\eta(1)} a_{\gamma(1)} \cdots a_{\gamma(m-i)} - a_{\eta(i)} \cdots a_{\eta(1)} a_{\gamma(1)} \cdots a_{\gamma(m-i)} a_m,$$

proceeding as in lemma 4.2.2 we may extend i to run up to $m-1$ and the $(m-1-i, i)$ shuffles to $(m-i, i)$ shuffles. Thus we have $[a_m [a_1 \cdots a_{m-1}]] = [a_1 \cdots a_m]$. As an alternative, induction on m also proves this result. $\square$

Lemma 4.2.5. *Let $w = rs$ be as above. Then $[\theta_j(w)] = 0$.*

Proof. This is obvious. □

Proposition 4.2.6. *Every Lie word, and therefore every Lie polynomial, is a linear combination of elements of the form* $[a_1 \cdots a_m]$. *More over,* $\|P\|_L = m$.

Proof. We proceed by induction on the order $\|P\|_L = m$ of a Lie word P. For $m = 1$ and $m = 2$ this is obvious. Suppose that this is also true for all $m < k$. Now for a Lie word P of order k, there exist two Lie words Q and R, of order $r < k$ and $s < k$ respectively such that $r+s = k$ and $P = [Q, R]$. Then, by the induction hypothesis we have that $Q = [a_1 \cdots a_r]$ and $R = [b_1 \cdots b_s]$. Thus $P = [[a_1 \cdots a_r], [b_1 \cdots b_s]] = [a_1 \cdots a_r [b_1 \cdots b_s]]$. Lemma 4.2.2 provides a decomposition of $[b_1 \cdots b_s]$ into a linear combination of concatenation of some permutation of the letters $b_1, \ldots, b_s$. The proposition follows. □

4.2.2 Permutations as Polynomials

As insightful as the previous results may be. We still are considering a very broad space. To recover our original space of matrix products, we need to consider first the subspaces of $\mathbb{C}\mathrm{w}(a, k)$ that are spanned by the words of length k that use the letters $\{1, \ldots, k\}$ exactly once. Let us denote by W_k these subspaces. The intersections of the Lie words and the Lie polynomials with W_k are denoted by L^k and LP^k respectively. It is clear that W^k can be identified with the free group vector space $\mathbb{C}\mathfrak{G}_k$ so that each permutation σ defines a word $\sigma(1) \cdots \sigma(k)$ just as we briefly mentioned in the introduction. Thus we need to find a basis for $\mathbb{C}\mathfrak{G}_k$, we do this by finding basis for complementary subspaces. First we focus on the subspace of Lie polynomials LP^k. For the rest of this work, k will be fixed.

Let (γ, η) be an $(l - i, i)$ shuffle for some $0 \leq i \leq l - 1$ and $l \leq k$. We define the permutation $\Psi(\gamma, \eta) \in \mathfrak{G}_k$ as

$$\Psi(\gamma,\eta) = \begin{pmatrix} 1 & \cdots & k-l & k-l+1 & \cdots & k-l+i & k-l+i+1 & \cdots & k \\ 1 & \cdots & k-l & k-l+\eta(i) & \cdots & k-l+\eta(1) & k-l+\gamma(1) & \cdots & k-l+\gamma(l-i) \end{pmatrix}.$$

.

Theorem 4.2.7. *Every Lie word* $P \in L^k$*, and therefore every Lie polynomial in* LP^k*, is a linear combination of elements of the form* $[\sigma(1) \cdots \sigma(k)]$*, where* $\sigma \in \mathfrak{G}_k$ *and* $\sigma(1) = 1$. *Moreover these elements are linearly independent and thus form a basis for* LP^k.

Proof. We proceed by induction. Clearly for $k = 1, 2$ this is true. Suppose it is true every positive integer less than k. We choose Lie words Q and R of orders r and s such that $P = [Q, R]$ and $r + s = k$. Then by linearity, it suffices to show that the expression

$$[[\sigma(1) \cdots \sigma(r)], [\sigma(r+1) \cdots \sigma(k)]]$$

is as in the statement of the theorem; where σ is a permutation of $\{1, \ldots r\}$ that fixes 1 and $r + 1$. By Proposition 4.2.1 this equals

$$-[\rho(r+1) \cdots \rho(k)[\sigma(1) \cdots \sigma(r)]],$$

which in turn, by Theorem 4.2.3 can be expanded as

$$P = \sum_{i=0}^{r-1} \sum_{(\gamma,\eta)} (-1)^i [[\sigma(1) \cdots \sigma(r)]\sigma(r+\eta(i)) \cdots \sigma(r+\eta(1))\sigma(r+\gamma(1)) \cdots \sigma(r-\gamma(s-i))].$$

Note that $r = k - s$; let us set $\mu = \Psi(\gamma, \eta)$. Now we may rewrite

$$P = \sum_{i=0}^{r-1} \sum_{\mu} (-1)^i [\sigma\mu(1) \cdots \sigma\mu(k)].$$

Since both σ and μ fix 1, our first assertion follows. To see that they are linearly independent note that for any $\sigma \in \mathfrak{G}_k$ that fixes 1. There is only one shuffle (γ, η) on the expansion of $[\sigma(1) \cdots \sigma(k)]$ described in Lemma 4.2.2 such that the resulting term is the word $\sigma(1) \cdots \sigma(k)$. Moreover, this is the only shuffle that fixes $\sigma(1) = 1$ as the first letter of the words in the expansion; thus no other word $\rho(1) \cdots \rho(k)$ for other permutation ρ is in such expansion. This implies the linear independence. ☐

4.3 Other symmetric sectors

To introduce other symmetric sectors we introduce a useful notation for partitions of integers as the decomposition here proposed will be given by the direct sum

$$\bigoplus_{\lambda \text{ partition of } k} V_k = \mathbb{C}\mathfrak{G}_k,$$

for a linear subspace V_λ associated with each partition λ of k. This new notation is given in the form of a definition.

Definition 4.3.1. A partition λ of an integer k is a k-tuple $(\lambda_1, \cdots \lambda_k)$ of integers, such that

$$\sum_{m=1}^{k} m\lambda_m = k.$$

To define the subspaces V_λ we write the basis for it explicitely. In orther to do so, first we define the symmetrizer, which is a linear map $S_m : A_1 \times \cdots \times A_m \to \mathbb{C}\mathfrak{G}_k$, where each A_i is a subspace of $\mathbb{C}\mathfrak{G}_k$ containing polynomials of order $\ell(i)$, such that $\sum_{i=1}^{m} \ell(i) = k$. It clearly suffices to define S_m acting on words. This is given by

$$S_m(w_1, \ldots, w_m) = \sum_{\sigma \in \mathfrak{G}_m} w_{\sigma(1)} \cdots w_{\sigma(m)}.$$

It follows that $S_1 : \mathbb{C}\mathfrak{G}_k \to \mathbb{C}\mathfrak{G}_k$ is the identity. Now we fix a partition λ of k and describe a basis β_λ for V_λ. Let $z(\lambda)$ be the number of non-zero entries in the k-tuple $\lambda = (\lambda_1, \ldots, \lambda_k)$. Let $\mathfrak{R}(\lambda) = \{r_1, \ldots, r_{z(\lambda)}\}$ be the collection of pairs $r = (i, \lambda_i)$ where λ_i is a non-zero entry of λ ordered by $r = (i, \lambda_i) < r' = (j, \lambda_j)$ iff $i < j$. For notational purposes we will write $r_i \in \mathfrak{R}(\lambda)$ as $r_i = (r_i^1, r_i^2) = (j, \lambda_j)$. Now we define $\mathfrak{W}(\lambda)$ as the collection of all $m(\lambda)$-tuples of words

$$\mathfrak{w} = \left(w_1^1, \ldots, w_{r_1^2}^1, w_1^2, \ldots, w_{r_2^2}^2, \ \ldots \ , w_1^{z(\lambda)}, \ldots, w_{r_{z(\lambda)}^2}^{z(\lambda)}\right),$$

where each word w_j^i is of length r_i^1 and so that no two words w_j^i and $w_{j'}^{i'}$ have any latter in common. We also impose that each word $w_i^j = a_1 \cdots a_{r_i^2}$ satisfies that $a_1 = \min\{a_1, \ldots, a_{r_i^2}\}$.

Proposition 4.3.2. *For a fixed partition λ of k. The integer $m(\lambda)$ as defined above is given by $\sum_i^{z(\lambda)} r_i^2$; and the concatenation of the words in each $\mathfrak{w} \in \mathfrak{W}(\lambda)$ is a word of order k.*

Proof. This follows easily from the definition. □

Finally, we set β_λ as the collection of all polynomial p of the form

$$p = S_{m(\lambda)}(\mathfrak{w}) \in \mathbb{C}\mathfrak{G}_k, \qquad\qquad \mathfrak{w} \in \mathfrak{W}(\lambda).$$

So defined it follows easily that $V_\lambda = \mathcal{L}$ for $\lambda = (0, \cdots, 0, 1)$. Finally we present the Mathematica implementation for the dimensions of the space V_λ, to check that they add to $k!$. We also show how this program calculates the basis for all sectors in the space $\mathbb{C}\mathfrak{G}_k$.

```
Parts[n_] := Module[{p, partitions, ones, k},
  p = Table[0, n];
  partitions = {};
  p[[n]] = 1;
  While[p[[1]] ≤ n,
   partitions = Join[partitions, {{p}}];
   ones = p[[1]];
   p[[1]] = 0;
   k = 2;
   While[p[[k]] == 0 && k < n, k++];
   If[ones == n, p[[1]] = n + 1];
   p[[k]] = p[[k]] - 1;
   p[[k - 1]] = p[[k - 1]] + 1;
   ones = ones + 1;
   While[ones > k - 1, p[[k - 1]] = p[[k - 1]] + 1; ones = ones - (k - 1)];
   p[[ones]] = p[[ones]] + 1;
  ];
  partitions
 ]

(*Some functions that make my life easier*)

checkpermitedperm[y_, x_] := If[y[[1]] == x[[1]], 1, 0];
IndLab[vars_, i_] := Module[{Choosing, IndependLabels, IndLabels},
   Choosing = Subsets[vars, {i}];
IndependLabels = {};
For[k = 1, k ≤ Length[Choosing], k++,
    x[k] = Choosing[[k]];
    y[k] = Permutations[x[k]];
    IndLabels = Select[y[k], checkpermitedperm[#, x[k]] == 1 &];
    IndependLabels = Join[IndependLabels, IndLabels]
   ];
   IndependLabels];
```

```
IndLabAll[list_, i_] := Module[{res},
  res = {};
  Do[ res = Join[res, IndLab[Subsets[list, {i}][[j]], i]] , {j, Length@ (Subsets[list, {i}])}];
  res]
F[i_, before_] := Module[{inds, res, stlabels, list},
  inds = {};
  Do[res = before[[j]];
   stlabels = DeleteCases[Labels, Alternatives @@ Flatten@res];
   list = IndLabAll[stlabels, i];
   at = {};
   Do[at = Join[at, {Join[{res}, {list[[k]]}]}], {k, Length@list}];
   inds = Join[inds, at]
   , {j, Length@before}];
  inds = (Level[#, {-2}] & /@ inds //. {} → Sequence[]);
  Union[Sort /@ inds]
 ]
(*Now we find all labeling of diagrams for each partition λ*)
Diagramsperpart[λ_, n_] := Module[{vars},
  vars = {{}};
  Do[
   Do[
     vars = F[i, vars];
     , {j, 1, λ[[i]], 1}];
   , {i, n, 1, -1}];
  vars
 ]
(*Now for all partitions*)
Diagrams[n_] := Module[{list},
  Labels = Table[i, {i, n}];
  list = {};
  Do[λ = Flatten@Parts[n][[j]];
   list = Join[list, Diagramsperpart[λ, n]]
   , {j, Length@Parts[n]}];
  list
 ]
```

```
In[2]:= DiagramForPartition[x_, n_] := Factorial[n] * Product[1 / (i ^ (x[[i]]) * Factorial[x[[i]]]), {i, n}]

In[3]:= Checkdimension[n_] := Plus @@ (DiagramForPartition[#, n] & @@@ Parts[n])

In[15]:=
        For[m = 2, m ≤ 50, m++,
         If[Checkdimension[m] == Factorial[m], Print["It works for n=", m], Print["WARNING! It does not work for n=", m]]
        ]

        It works for n=2
        It works for n=3
        It works for n=4
        It works for n=5
        It works for n=6
        It works for n=7
        It works for n=8


        It works for n=40
        It works for n=41
        It works for n=42
        It works for n=43
        It works for n=44
        It works for n=45
        It works for n=46
        It works for n=47
        It works for n=48
        It works for n=49

Out[15]= $Aborted
```

Bibliography

[Alekseev and Ivanov, 2016] Alekseev, I. and Ivanov, S. O. (2016). Higher Jacobi identities. *arXiv:1604.05281 [math]*. arXiv: 1604.05281.

[Bourbaki, 1975] Bourbaki, N. (1975). *Lie groups and lie algebras*. Springer.

[Fulton and Harris, 2004] Fulton, W. and Harris, J. (2004). *Representation theory: a first course*. Springer.

[Hentschinski, 2012] Hentschinski, M. (2012). Pole prescription of higher order induced vertices in Lipatov's QCD effective action. *Nuclear Physics B*, 859(2):129–142.

[Lipatov, 1995] Lipatov, L. N. (1995). Gauge invariant effective action for high energy processes in QCD. *Nuclear Physics B*, 452(1):369–397.

[Reutenauer, 1993] Reutenauer, C. (1993). *Free Lie Algebras*. Clarendon Press.

[Zee, 2016] Zee, A. (2016). *Group theory in a nutshell for physicists*. Princeton University Press.