Chapter 3

The scalar field in BF gravity

The main purpose of this thesis is to study an action principle that will succesfully couple a scalar field to gravity in the framework of the CMPR action principle. We take on this duty on this chapter.

The action of the BF type for gravity with cosmological constant and the Immirzi parameter has the following form [11]:

$$S_G[B,A,\psi,\mu] = \int_{\mathcal{M}^4} [B^{IJ}\wedge F_{IJ}[A] - \frac{1}{2}\psi_{IJKL}B^{IJ}\wedge B^{KL} - \mu(a_1\psi_{IJ}{}^{IJ} + a_2\psi_{IJKL}\varepsilon^{IJKL} - \mathcal{H}) + l_1 B_{IJ}\wedge B^{IJ} + l_2 B_{IJ}\wedge *B^{IJ}]. \tag{3.1}$$

We would like to study an action for the scalar field, φ, in terms of the dynamical variables of the previous equation. The action proposed is:

$$S_\varphi[B,\varphi,\pi] = \int_{\mathcal{M}^4} \big[(m_1 B_{IJ}\wedge B^{IJ} + m_2 B_{IJ}\wedge *B^{IJ})\pi^\mu\partial_\mu\varphi + (m_3\varepsilon^{\alpha\beta\gamma\delta}B^{IJ}_{\mu\alpha}B_{\beta\gamma J}{}^{K}B_{\delta\nu KI} + m_4\varepsilon^{\alpha\beta\gamma\delta}B^{IJ}_{\mu\alpha}B_{\beta\gamma J}{}^{K} * B_{\delta\nu KI})\pi^\mu\pi^\nu d^4x\big], \tag{3.2}$$

where the four m_i are coupling constants, B^{IJ} is the antisymmetric 2-form $B^{IJ} = \alpha^*(e^I \wedge e^J) + \beta e^I \wedge e^J$ as defined in the CMPR action, and π^μ is a Lagrange multiplier. By the expanded form of the dynamical field, $B^{IJ}_{\mu\nu}$, we mean

$$B^{IJ}_{\mu\nu} dx^\mu \wedge dx^\nu = \left[\frac{\alpha}{2}\varepsilon^{IJ}{}_{KL} e^K_\mu e^L_\nu + \beta e^I_\mu e^J_\nu\right] dx^\mu \wedge dx^\nu. \tag{3.3}$$

The action principle in Eq. (3.2) makes sense because the first two terms (the ones multiplying m_1 and m_2) are simply the two natural volume terms of the theory, which play the role of $\sqrt{-g}d^4x$. As for the terms which multiply m_3 and m_4, we have learned from [12] that the metric density is *cubic* in the field variables of BF gravity. This identity holds in terms of the B^{IJ} of CMPR, but we also add a cubic term for the dual of the dynamical variable, in correspondance with the second volume term $B_{IJ} \wedge *B^{IJ}$, the one that multiplies the constant m_2.

We will verify that this action reduces effectively to the usual scalar field action (2.6). The variation with respect to the Lagrange multiplier is

$$\begin{aligned} 0 = S_\varphi[B, \varphi, \pi + \delta\pi] - S_\varphi[B, \varphi, \pi] = \int_{\mathcal{M}^4} (m_1 B_{IJ} \wedge B^{IJ} \\ + m_2 B_{IJ} \wedge *B^{IJ})\delta\pi^\mu \partial_\mu\varphi + (m_3 \varepsilon^{\alpha\beta\gamma\delta} B^{IJ}_{\mu\alpha} B_{\beta\gamma J}{}^K B_{\delta\nu KI} \\ + m_4 \varepsilon^{\alpha\beta\gamma\delta} B^{IJ}_{\mu\alpha} B_{\beta\gamma J}{}^K * B_{\delta\nu KI})(\delta\pi^\mu\pi^\nu + \pi^\mu\delta\pi^\nu) d^4x], \end{aligned}$$

so we have the following equation of motion,

$$\begin{aligned} \delta\pi : (m_1 B_{IJ} \wedge B^{IJ} + m_2 B_{IJ} \wedge *B^{IJ})\partial_\mu\varphi \\ +2(m_3 \varepsilon^{\alpha\beta\gamma\delta} B^{IJ}_{(\mu|\alpha} B_{\beta\gamma J}{}^K B_{\delta|\nu)KI} + m_4 \varepsilon^{\alpha\beta\gamma\delta} B^{IJ}_{(\mu|\alpha} B_{\beta\gamma J}{}^K * B_{\delta|\nu)KI})\pi^\nu d^4x = 0. \end{aligned} \tag{3.4}$$

We would like to express the equation of motion of π^μ in terms of the expanded 2-forms, so we use Eq. (3.3) to rewrite Eq. (3.4) as

$$d^4x\varepsilon^{\alpha\beta\gamma\delta}[(m_1 B^{IJ}_{\alpha\beta}B_{\delta\gamma IJ} + m_2 B^{IJ}_{\alpha\beta} * B_{\delta\gamma IJ})\partial_\mu\varphi$$
$$+2(m_3 B^{IJ}_{(\mu|\alpha}B_{\beta\gamma J}{}^K B_{\delta|\nu)KI} + m_4 B^{IJ}_{(\mu|\alpha}B_{\beta\gamma J}{}^{K*}B_{\delta|\nu)KI})\pi^\nu] = 0, \qquad (3.5)$$

where we have written the Levi-Civita symbol that carries the antisymmetry of the wedge products that multiply the coordinate basis on the terms that multiply m_1 and m_2 in (3.4), explicitly:

$$dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = \varepsilon^{\mu\nu\rho\sigma} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = \varepsilon^{\mu\nu\rho\sigma} d^4x.$$

The next step is to substitute in the form of the B^{IJ} field in terms of the tetrad field. The calculations contain many terms, but become simplified after a few steps. For this reason, we wish to analyse Eq. (3.5) by parts. The term multiplying m_1 is:

$$\begin{aligned}
B^{IJ}_{\alpha\beta}B_{\delta\gamma IJ} &= (\frac{\alpha}{2}\varepsilon^{IJ}{}_{KL}e^K_{[\alpha}e^L_{\beta]} + \beta e^I_{[\alpha}e^J_{\beta]})(\frac{\alpha}{2}\varepsilon_{IJMN}e^M_{[\gamma}e^N_{\delta]} + \beta e_{[\gamma|I|}e_{\delta]J}) \\
&= \frac{\alpha^2}{4}\varepsilon^{IJKL}\varepsilon_{IJMN}e_{[\alpha|K|}e_{\beta]L}e^M_{[\gamma}e^N_{\delta]} + \frac{\alpha\beta}{2}\varepsilon_{IJKL}e^K_{[\alpha}e^L_{\beta]}e^I_{[\gamma}e^J_{\delta]} \\
&+ \frac{\alpha\beta}{2}\varepsilon_{IJMN}e^I_{[\alpha}e^J_{\beta]}e^M_{[\gamma}e^N_{\delta]} + \beta^2 e^I_{[\alpha}e^J_{\beta]}e_{[\gamma|I|}e_{\delta]J} \\
&= \alpha^2\sigma\delta^K_{[M}\delta^L_{N]}e_{[\alpha|K|}e_{\beta]L}e^M_{[\gamma}e^N_{\delta]} + \alpha\beta\varepsilon_{IJKL}e^I_{[\alpha}e^J_{\beta]}e^K_{[\gamma}e^L_{\delta]} \\
&+ \beta^2 e^I_{[\alpha}e^J_{\beta]}e_{[\gamma|I|}e_{\delta]J}.
\end{aligned}$$

Notice that we have explicitly shown the antisymmetries in the spacetime indices. It also a good idea to point out that $e_{\mu I} = \eta_{IJ}e^J_\mu$. Following our study of the first term, we have that

$$
\begin{aligned}
d^4x\varepsilon^{\alpha\beta\gamma\delta}B^{IJ}_{\alpha\beta}B_{\delta\gamma IJ} &= dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta \big[(\alpha^2+\beta^2)e_{[\alpha|M|}e_{\beta]N}e^M_{[\gamma}e^N_{\delta]} \\
&\quad + \alpha\beta\varepsilon_{IJKL}e^I_{[\alpha}e^J_{\beta]}e^K_{[\gamma}e^L_{\delta]}\big] \\
&= (\alpha^2\sigma+\beta^2)e_M \wedge e_N \wedge e^M \wedge e^N \\
&\quad + \varepsilon_{IJKL}\alpha\beta e^I \wedge e^J \wedge e^K \wedge e^L \\
&= \alpha\beta\varepsilon_{IJKL}e^I \wedge e^J \wedge e^K \wedge e^L. \qquad (3.6)
\end{aligned}
$$

In this expression we have made use of the antisymmetric property of the wedge product, that only lets one term survive. We also switched back to a non-coordinate basis using the inverse tetrad relation $e^I = e^I_\mu dx^\mu$.

The second term can be computed similarly, since it is the dual counterpart of the first term. Multiplying m_2 we have

$$
\begin{aligned}
B^{IJ}_{\alpha\beta}B_{\delta\gamma IJ} &= (\frac{\alpha}{2}\varepsilon^{IJ}{}_{KL}e^K_{[\alpha}e^L_{\beta]} + \beta e^I_{[\alpha}e^J_{\beta]})(\alpha\sigma e_{[\gamma|I|}e_{\delta]J} + \frac{\beta}{2}\varepsilon_{IJMN}e^M_{[\gamma}e^N_{\delta]}) \\
&= \frac{\alpha^2\sigma}{2}\varepsilon_{IJKL}e^K_{[\alpha}e^L_{\beta]}e^I_{[\gamma}e^J_{\delta]} + \frac{\alpha\beta}{4}\varepsilon^{IJ}{}_{KL}e^K_{[\alpha}e^L_{\beta]}\varepsilon_{IJMN}e^M_{[\gamma}e^N_{\delta]} \\
&\quad + \alpha\beta\sigma e^I_{[\alpha}e^J_{\beta]}e_{[\gamma|I|}e_{\delta]J} + \frac{\beta^2}{2}\varepsilon_{IJMN}e^I_{[\alpha}e^J_{\beta]}e^M_{[\gamma}e^N_{\delta]}.
\end{aligned}
$$

So together with the Levi-Civita symbol,

$$
\begin{aligned}
d^4x\varepsilon^{\alpha\beta\gamma\delta}B^{IJ}_{\alpha\beta}B_{\delta\gamma IJ} &= dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta \big[\frac{\alpha^2\sigma+\beta^2}{2}\varepsilon_{IJKL}e^I_{[\alpha}e^J_{\beta]}e^K_{[\gamma}e^L_{\delta]} \\
&\quad + 2\alpha\beta\sigma e^I_{[\alpha}e^J_{\beta]}e_{[\gamma|I|}e_{\delta]J}\big] \\
&= \frac{\alpha^2\sigma+\beta^2}{2}\varepsilon_{IJKL}e^I \wedge e^J \wedge e^K \wedge e^L + 2\alpha\beta e^I \wedge e^J \wedge e_I \wedge e_J \\
&= \frac{\alpha^2\sigma+\beta^2}{2}\varepsilon_{IJKL}e^I \wedge e^J \wedge e^K \wedge e^L. \qquad (3.7)
\end{aligned}
$$

Not surprisingly, the symmetries killed the term that survived in (3.6) and, instead, the remaining term was the one that was equal to zero in the non-dual case.

The computation of the third term requires special care. The steps are the following for the term multiplying m_3:

$$
\begin{aligned}
&B^{IJ}_{(\mu|\alpha}B_{\beta\gamma J}{}^{K}B_{\delta|\nu)KI} = (\frac{\alpha}{2}\varepsilon^{IJ}{}_{MN}e^{M}_{[\mu}e^{N}_{\alpha]} + \beta e^{I}_{[\mu}e^{J}_{\alpha]})\\
&(\frac{\alpha}{2}\varepsilon_{J}{}^{K}{}_{PQ}e^{P}_{[\beta}e^{Q}_{\gamma]} + \beta e_{[\beta|J|}e^{K}_{\gamma]})(\frac{\alpha}{2}\varepsilon_{KIRS}e^{R}_{[\delta}e^{S}_{\nu]} + \beta e_{[\delta|K|}e_{\nu]I})\\
&= \frac{\alpha^3}{8}\varepsilon^{IJ}{}_{MN}\varepsilon_{J}{}^{K}{}_{PQ}\varepsilon_{KIRS}e^{M}_{[\mu}e^{N}_{\alpha]}e^{P}_{[\beta}e^{Q}_{\gamma]}e^{R}_{[\delta}e^{S}_{\nu]}\\
&+ \frac{\alpha^2\beta}{4}\big(\varepsilon^{IJMN}\varepsilon_{JKPQ}e_{[\mu|M|}e_{\alpha]N}e^{P}_{[\beta}e^{Q}_{\gamma]}e^{K}_{[\delta}e_{\nu]I}\\
&+ \varepsilon^{IJMN}\varepsilon_{KIRS}e_{[\mu|M|}e_{\alpha]N}e_{[\beta|J|}e^{K}_{\gamma]}e^{R}_{[\delta}e^{S}_{\nu]} + \varepsilon_{J}{}^{K}{}_{PQ}\varepsilon_{KIRS}e^{I}_{[\mu}e^{J}_{\alpha]}e^{P}_{[\beta}e^{Q}_{\gamma]}e^{R}_{[\delta}e^{S}_{\nu]}\big)\\
&+ \frac{\alpha\beta^2}{2}\big(\varepsilon^{IJ}{}_{MN}e^{M}_{[\mu}e^{N}_{\alpha]}e_{[\beta|J|}e^{K}_{\gamma]}e_{[\delta|K|}e_{\nu]I} + \varepsilon_{J}{}^{K}{}_{PQ}e^{I}_{[\mu}e^{J}_{\alpha]}e^{P}_{[\beta}e^{Q}_{\gamma]}e_{[\delta|K|}e_{\nu]I}\\
&+ \varepsilon_{KIRS}e^{I}_{[\mu}e^{J}_{\alpha]}e_{[\beta|J|}e^{K}_{\gamma]}e^{R}_{[\delta}e^{S}_{\nu]}\big) + \beta^3 e^{I}_{[\mu}e^{J}_{\alpha]}e_{[\beta|J|}e^{K}_{\gamma]}e_{[\delta|K|}e_{\nu]I}\\
&= -\frac{3\alpha^3\sigma}{4}\varepsilon_{J}{}^{K}{}_{PQ}\delta^{J}_{[K}\delta^{M}_{E}\delta^{N}_{F]}e_{[\mu|M|}e_{\alpha]N}e^{P}_{[\beta}e^{Q}_{\gamma]}e^{E}_{[\delta}e^{F}_{\nu]}\\
&- \frac{3\alpha^2\beta\sigma}{2}\big(\delta^{I}_{[K}\delta^{M}_{P}\delta^{N}_{Q]}e_{[\mu|M|}e_{\alpha]N}e^{P}_{[\beta}e^{Q}_{\gamma]}e^{K}_{[\delta}e_{\nu]I}\\
&+ \delta^{J}_{[K}\delta^{M}_{R}\delta^{N}_{S]}e_{[\mu|M|}e_{\alpha]N}e_{[\beta|J|}e^{K}_{\gamma]}e^{R}_{[\delta}e^{S}_{\nu]} + \delta^{J}_{[I}\delta^{P}_{R}\delta^{Q}_{S]}e^{I}_{[\mu}e_{\alpha]J}e_{[\beta|P|}e_{\gamma]Q}e^{R}_{[\delta}e^{S}_{\nu]}\big)\\
&+ \frac{\alpha\beta^2}{2}\big(\varepsilon^{IJ}{}_{MN}e^{M}_{[\mu}e^{N}_{\alpha]}e_{[\beta|J|}e^{K}_{\gamma]}e_{[\delta|K|}e_{\nu]I} + \varepsilon_{J}{}^{K}{}_{PQ}e^{I}_{[\mu}e^{J}_{\alpha]}e^{P}_{[\beta}e^{Q}_{\gamma]}e_{[\delta|K|}e_{\nu]I}\\
&+ \varepsilon_{KIRS}e^{I}_{[\mu}e^{J}_{\alpha]}e_{[\beta|J|}e^{K}_{\gamma]}e^{R}_{[\delta}e^{S}_{\nu]}\big) + \beta^3 e^{I}_{[\mu}e^{J}_{\alpha]}e_{[\beta|J|}e^{K}_{\gamma]}e_{[\delta|K|}e_{\nu]I}.
\end{aligned}
$$

where in the last step we performed a contraction of the Levi-Civita tensor densities. Together with the Levi-Civita symbol that multiplies m_3 in Eq. (3.5) we have that the only surviving term is

$$
\begin{aligned}
d^4x\varepsilon^{\alpha\beta\gamma\delta}B^{IJ}_{(\mu|\alpha}B_{\beta\gamma J}{}^{K}B_{\delta|\nu)KI} &= dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}\wedge dx^{\delta}B^{IJ}_{(\mu|\alpha}B_{\beta\gamma J}{}^{K}B_{\delta|\nu)KI}\\
&= \frac{\alpha\beta^2}{2}\varepsilon_{MNPQ}e_{I\mu}e^{I}_{\mu}e^{M}\wedge e^{N}\wedge e^{P}\wedge e^{Q}, \quad (3.8)
\end{aligned}
$$

in account for the symmetries of the terms. Once again, we have expressed Eq. (3.8) in terms of a non-coordinate basis.

The computation of the fourth and last term is similar to that one of the third term. Multiplying m_4 we have

$$
\begin{aligned}
&B^{IJ}_{(\mu|\alpha}B_{\beta\gamma J}{}^{K} * B_{\delta|\nu)KI} = (\frac{\alpha}{2}\varepsilon^{IJ}{}_{MN}e^{M}_{[\mu}e^{N}_{\alpha]} + \beta e^{I}_{[\mu}e^{J}_{\alpha]})\\
&(\frac{\alpha}{2}\varepsilon_{J}{}^{K}{}_{PQ}e^{P}_{[\beta}e^{Q}_{\gamma]} + \beta e_{[\beta|J|}e^{K}_{\gamma]})(\alpha\sigma e_{[\delta|K|}e_{\nu]I} + \frac{\beta}{2}\varepsilon_{KIRS}e^{R}_{[\delta}e^{S}_{\nu]})\\
&= \frac{\alpha^3\sigma}{4}\varepsilon^{IJ}{}_{MN}\varepsilon_{J}{}^{K}{}_{PQ}e^{M}_{[\mu}e^{N}_{\alpha]}e^{P}_{[\beta}e^{Q}_{\gamma]}e_{[\delta|K|}e_{\nu]I}\\
&+ \frac{\alpha^2\beta}{2}\big(\frac{1}{4}\varepsilon^{IJ}{}_{MN}\varepsilon_{J}{}^{K}{}_{PQ}\varepsilon_{KIRS}e^{M}_{[\mu}e^{N}_{\alpha]}e^{P}_{[\beta}e^{Q}_{\gamma]}e^{R}_{[\delta}e^{S}_{\nu]}\\
&+ \sigma\varepsilon^{IJ}{}_{MN}e^{M}_{[\mu}e^{N}_{\alpha]}e_{[\beta|J|}e^{K}_{\gamma]}e_{[\delta|K|}e_{\nu]I} + \sigma\varepsilon_{J}{}^{K}{}_{PQ}e^{I}_{[\mu}e^{J}_{\alpha]}e^{P}_{[\beta}e^{Q}_{\gamma]}e_{[\delta|K|}e_{\nu]I}\big)\\
&+ \frac{\alpha\beta^2}{4}\big(\varepsilon^{IJ}{}_{MN}\varepsilon_{KIRS}e^{M}_{[\mu}e^{N}_{\alpha]}e_{[\beta|J|}e^{K}_{\gamma]}e^{R}_{[\delta}e^{S}_{\nu]}\\
&+ \varepsilon_{J}{}^{K}{}_{PQ}\varepsilon_{KIRS}e^{I}_{[\mu}e^{J}_{\alpha]}e^{P}_{[\beta}e^{Q}_{\gamma]}e^{R}_{[\delta}e^{S}_{\nu]}\\
&+ 4\sigma e^{I}_{[\mu}e^{J}_{\alpha]}e_{[\beta|J|}e^{K}_{\gamma]}e_{[\delta|K|}e_{\nu]I}\big) + \frac{\beta^3}{2}\varepsilon_{KIRS}e^{I}_{[\mu}e^{J}_{\alpha]}e_{[\beta|J|}e^{K}_{\gamma]}e^{R}_{[\delta}e^{S}_{\nu]}\\
&= -\frac{3}{2}\alpha^3\delta^{I}_{[K}\delta^{M}_{P}\delta^{N}_{Q]}e_{[\mu|M|}e_{\alpha]N}e^{P}_{[\beta}e^{Q}_{\gamma]}e^{K}_{[\delta}e_{\nu]I}\\
&+ \frac{\alpha^2\beta}{2}\big(-\frac{3}{2}\sigma\varepsilon^{K}{}_{IRS}\delta^{I}_{[K}\delta^{M}_{P}\delta^{N}_{Q]}e^{M}_{[\mu}e^{N}_{\alpha]}e^{P}_{[\beta}e^{Q}_{\gamma]}e^{R}_{[\delta}e^{S}_{\nu]}\\
&+ \sigma\varepsilon^{IJ}{}_{MN}e^{M}_{[\mu}e^{N}_{\alpha]}e_{[\beta|J|}e^{K}_{\gamma]}e_{[\delta|K|}e_{\nu]I} + \sigma\varepsilon_{J}{}^{K}{}_{PQ}e^{I}_{[\mu}e^{J}_{\alpha]}e^{P}_{[\beta}e^{Q}_{\gamma]}e_{[\delta|K|}e_{\nu]I}\big)\\
&+ \frac{\alpha\beta^2}{2}\big(-3\sigma\delta^{J}_{[K}\delta^{M}_{E}\delta^{N}_{F}e_{[\mu|M|}e_{\alpha]N}e_{[\beta J}e^{K}_{\gamma]}e^{R}_{[\delta}e^{S}_{\nu}\\
&- 3\sigma\delta^{J}_{[I}\delta^{P}_{R}\delta^{Q}_{S}e^{I}_{[\mu}e_{\alpha]J}e_{[\beta|P|}e_{\gamma]Q}e^{R}_{[\delta}e^{S}_{\nu]}\\
&+ 2\sigma e^{I}_{[\mu}e^{J}_{\alpha]}e_{[\beta|J|}e^{K}_{\gamma]}e_{[\delta|K|}e_{\nu]I}\big) + \frac{\beta^3}{2}\varepsilon_{KIRS}e^{I}_{[\mu}e^{J}_{\alpha]}e_{[\beta|J|}e^{K}_{\gamma]}e^{R}_{[\delta}e^{S}_{\nu]}.
\end{aligned}
$$

So we have that, on account of the symmetries, the non-zero terms are

$$d^4x\varepsilon^{\alpha\beta\gamma\delta}B^{IJ}_{(\mu|\alpha}B_{\beta\gamma J}{}^{K} * B_{\delta|\nu)KI} = dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta$$
$$B^{IJ}_{(\mu|\alpha}B_{\beta\gamma J}{}^{K} * B_{\delta|\nu)KI}$$
$$= \frac{\alpha^2\beta\sigma}{2}\varepsilon_{MNPQ}e_{I\mu}e^I_\mu e^M \wedge e^N \wedge e^P \wedge e^Q. \tag{3.9}$$

Having computed all four terms [Equations (3.6), (3.7), (3.8), and (3.9)], the proposed scalar field action in Eq. (3.2) can be written as

$$S_\varphi[e,\varphi,\pi] = \int_{\mathcal{M}^4}[\frac{1}{2}[2m_1\alpha\beta + m_2(\alpha^2\sigma+\beta^2)]\pi^\mu(\partial_\mu\varphi) + \frac{1}{2}[m_3\alpha\beta^2$$
$$+ m_4\alpha^2\beta\sigma]e^I_\mu e_{I\nu}\pi^\mu\pi^\nu]\varepsilon_{JKLM}e^J \wedge e^K \wedge e^L \wedge e^M. \tag{3.10}$$

On the same line, the variation of π can be written in a compact way in terms of the tetrad field.

$$\delta\pi : [\big(m_1\alpha\beta + m_2(\alpha^2\sigma+\beta)\big)\,(\partial_\mu\varphi) + [m_3\alpha\beta^2$$
$$+m_4\alpha^2\beta\sigma]e^I_\mu e_{I\nu}\pi^\nu]\varepsilon_{JKLM}e^J \wedge e^K \wedge e^L \wedge e^M = 0, \tag{3.11}$$

which implies that

$$[m_1\alpha\beta + m_2(\alpha^2\sigma+\beta)](\partial_\mu\varphi) = -[m_3\alpha\beta^2 + m_4\alpha^2\beta\sigma]\eta_{IJ}e^I_\mu e^J_\nu\pi^\nu. \tag{3.12}$$

From this equation, we are ready to solve for the Lagrange multiplier π^μ:

$$\pi^\nu = -\frac{1}{2}\frac{2m_1\alpha\beta + m_2(\alpha^2\sigma+\beta^2)}{m_3\alpha\beta^2 + m_4\alpha^2\beta\sigma}\eta^{IJ}e^\mu_I e^\nu_J\partial_\mu\varphi. \tag{3.13}$$

Having computed the form of the multiplier, the job is almost complete.

Let us feed the information contained in the multiplier into equation (3.23).

$$\begin{aligned} S_\varphi[e,\varphi] &= \int\limits_{\mathcal{M}^4} \frac{1}{4}\frac{[2m_1\alpha\beta + m_2(\alpha^2\sigma+\beta^2)]^2}{m_3\alpha\beta^2 + m_4\alpha^2\beta\sigma}[-\eta^{IJ}e^\mu_I e^\nu_J(\partial_\mu\varphi)(\partial_\nu\varphi) \\ &+ \frac{1}{2}\eta_{IJ}e^I_\mu e^J_\nu \eta^{IK} e^\mu_I e^\rho_K \eta^{JL} e^\nu_J e^\sigma_L(\partial_\rho\varphi)(\partial_\sigma\varphi)]\varepsilon_{PQRS}e^P\wedge e^Q\wedge e^R\wedge e^S \\ &= \int\limits_{\mathcal{M}^4} \frac{1}{4}\frac{[2m_1\alpha\beta + m_2(\alpha^2\sigma+\beta^2)]^2}{m_3\alpha\beta^2 + m_4\alpha^2\beta\sigma}[-\eta^{IJ}e^\mu_I e^\nu_J(\partial_\mu\varphi)(\partial_\nu\varphi) \\ &+ \frac{1}{2}\delta^K_J \eta^{JL} e^\rho_K e^\sigma_L(\partial_\rho\varphi)(\partial_\sigma\varphi)]\varepsilon_{PQRS}\varepsilon^{\alpha\beta\gamma\delta}e^P_\alpha e^Q_\beta e^R_\gamma e^S_\delta d^4x, \end{aligned} \tag{3.14}$$

where we have canceled out the tetrad fields that appear together with their inverse in the second term on the first line, e.g. $e^I_\mu e^\mu_I = 1$, and contracted the covariant and (inverse) contravariant form of the Minkowski metric η_{IJ}, into a Kronecker delta on the second line.

We would like to see this equation in terms of the metric tensor, rather than the tetrad field, so we can compare it to the usual scalar field action introduced in Chapter 2 in equation (2.6). In order to achieve this, we have to remember that the tetrad is defined as [3]

$$g(e_I, e_J) = g_{\mu\nu}e^\mu_I e^\nu_J = \eta_{IJ}. \tag{3.15}$$

For this reason, by the way, e^I_μ is usually referred to as the *square root* of the metric in a colloquial sense. Furthermore, this definition allows us to relate the determinant of the metric tensor, g, to the determinant of the tetrad field,

$$g = \det(g_{\mu\nu}) = \det(e^I_\mu e^J_\nu \eta_{IJ}) = \sigma[\det(e^I_\mu)]^2. \tag{3.16}$$

On the other hand, the determinant of some $n\times n$ square matrix, A, obeys the identity

$$\det(A^\mu_\nu) = \varepsilon^{\alpha_1\alpha_2\ldots\alpha_n}A^1_{\alpha_1}A^2_{\alpha_2}, A^n_{\alpha_n}, \tag{3.17}$$

which allows us to calculate the determinant of the tetrad with some algebraic manipulation. We have that

$$\det(e^I_\nu) = \frac{1}{4!}\varepsilon\varepsilon_{IJKL}e^I_\alpha e^J_\beta e^K_\mu e^L_\nu \varepsilon^{\alpha\beta\mu\nu}. \tag{3.18}$$

where $\varepsilon = 1$ in our convention.Using equations (3.15), (3.16), and (3.18), we can write down Eq. (3.14) in terms of the metric tensor and the scalar field:

$$S_\varphi[g,\varphi] = -3!\frac{[2m_1\alpha\beta + m_2(\alpha^2\sigma + \beta^2)]^2}{m_3\alpha\beta^2 + m_4\alpha^2\beta\sigma}\int\limits_{\mathcal{M}^4}\frac{1}{2}g^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi)\sqrt{-g}d^4x. \tag{3.19}$$

This is precisely the usual scalar field action, as in equation (2.6), multiplied by a constant factor. It cannot be ignored that Eq. (3.19) also dictates the correct relationship between the four coupling constants, m_i. The relation is

$$3![2m_1\alpha\beta + m_2(\alpha^2\sigma + \beta^2)]^2 = m_3\alpha\beta^2 + m_4\alpha^2\beta\sigma. \tag{3.20}$$

So equation (3.19) is the scalar field action in the CMPR framework with the relation (3.20) determining a family of scalar field actions from which one can recover the old action on the right scale.

It is clear that the equations of motion fall immediately once the coupling constants have been correcly chosen in accordance with Eq. (3.20).

Having pointed out the relevance of the tetrad, it is impossible to resist the temptation of writing Eq. (3.2) in terms of this field. While in principle Eq. (3.14) does the job, it is possible to write down a cleaner expression. We simply have to work our way backwards from equation (3.19) and take into account the relation of Eq. (3.20). We find that

$$S_\varphi[e,\varphi] = -\frac{1}{2}\int\limits_{\mathcal{M}^4} \det(e^I_\mu)\,\eta^{IJ}\,e^\mu_I\,e^\nu_J\,(\partial_\mu\varphi)\,(\partial_\nu\varphi)\,d^4x. \tag{3.21}$$

Final remarks are made in the Conclusions Chapter.