Chapter 2

The scalar field in General Relativity

The purpose of this work is to perform the coupling of the scalar field to BF gravity, it is therefore an important matter to discuss how the scalar field is usually coupled in the context of the general theory of relativity.

The full form of the Einstein field equations is

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.1}$$

with $T_{\mu\nu}$, the energy-momentum tensor, being a conserved quantity, such that $\nabla_\mu T^{\mu\nu} = 0$, that describes the matter effects in field theory, and G Newton's gravitational constant. The factor that determines the intensity of the gravitational effects, $8\pi G$, is determined by matching the Einstein field equations in the Newtonian weak field limit, in which $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the second term being a small perturbation that makes the metric tensor deviate from flat spacetime.

It is possible to obtain the Einstein field equations from a variational principle. The general form of the action is

$$S = S_G + S_M, \tag{2.2}$$

with S_G the vacuum contribution, which may as well be decomposed into a gravitational part and the cosmological term, $S_G = S_g + S_\Lambda$, and S_M is the matter contribution, related to the energy-momentum tensor like [5]

$$T_{\mu\nu} \equiv -2\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{2.3}$$

Following this convention, the action principles that we studied in vacuum should be normalized like $S_G \to \frac{1}{16\pi G}S_G$ to account for the $8\pi G$ factor in the Einstein field equations.

We are currently concerned with the coupling of the scalar field. The energy momentum tensor associated with it is

$$T_{(\varphi)\mu\nu} = \nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\nabla_\rho\varphi\nabla_\sigma\varphi - g_{\mu\nu}V[\varphi]. \tag{2.4}$$

We call φ the scalar field and $V[\varphi]$ is some potential that depends on the scalar field.

Let us derive an action principle for φ that will yield the energy momentum tensor of Eq. (2.4). This task is straightforward and can be found in the literature. The canonical Lagrangian from classical mechanics can be rewritten in the context of classical field theory [5]:

$$L = \frac{1}{2}m\dot{q}^2 - V(q) \to \mathcal{L} = \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}[\mathrm{grad}(\varphi)]^2 - V(\varphi).$$

In the former expresion the first term on the right hand side corresponds to the kinetic energy, the second term is the gradient energy, and the third term is the usual potential energy. The next step is to rewrite the Lagrangian density, $\mathcal{L}$, by taking the advantage of the fact that the Minkowski metric $\eta_{\mu\nu}$ is diagonal, $\eta_{\mu\nu} = \mathrm{diag}(\sigma, 1, 1, 1)$, with $\sigma = -1$:

$$\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}[\mathrm{grad}(\varphi)]^2 - V(\varphi) = -\frac{1}{2}\eta^{IJ}(\partial_I\varphi)(\partial_J\varphi) - V(\varphi).$$

Finally, having constructed a Lagrangian on a local frame, one can use the *colon goes to semicolon* rule to write down the Lagrangian in a fully covariant form:

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} (\nabla_\mu \varphi)(\nabla_\nu \varphi) - V(\varphi). \tag{2.5}$$

So the scalar field action is

$$S_\varphi = - \int\limits_{\mathcal{M}^4} d^4x \, \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} (\partial_\mu \varphi)(\partial_\nu \varphi) + V(\varphi) \right]. \tag{2.6}$$

The square root of the negative determinant of the metric, $\sqrt{-g}$ comes into play to account for the fact that Eq. (2.5) is a Lagrangian density. Let us show how the variation of this action principle with respect to the inverse metric is:

$$\begin{aligned} \delta S_\varphi = &-\frac{1}{2} \int\limits_{\mathcal{M}^4} d^4x \, \sqrt{-g} (\partial_\mu \varphi)(\partial_\nu \varphi) \delta g^{\mu\nu} \\ &+ \int\limits_{\mathcal{M}^4} d^4x \, \left[\frac{1}{2} g^{\alpha\beta} (\partial_\alpha \varphi)(\partial_\beta \varphi) + V(\varphi) \right] \left(\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \right). \end{aligned} \tag{2.7}$$

So indeed the energy momentum tensor of Eq. (2.4) is recovered by using the definition (2.3).

Let us study another action principle for the scalar field. The reason for this, is that the form of this Lagrangian most closely resembles the scalar field Lagrangian in the BF theory. For simplicity, we can make $V[\varphi] = 0$. Let

$$S_\varphi[g, \varphi, \pi] = \int\limits_{\mathcal{M}^4} d^4x \, \sqrt{-g} \left[\pi^\mu \partial_\mu \varphi - \frac{1}{2} g_{\mu\nu} \pi^\mu \pi^\nu \right], \tag{2.8}$$

where π^μ is a Lagrange multiplier. We can show that this action is equivalent to (2.6) by computing the equation of motion for the π^μ:

$$\delta\pi : g^{\mu\nu} (\partial_\nu \varphi) - \pi^\mu = 0. \tag{2.9}$$

Plugging this equation of motion into Eq. (2.8) results in

$$S_\varphi = -\int_{\mathcal{M}^4} d^4x\,\sqrt{-g}\frac{1}{2}g^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi), \tag{2.10}$$

which is equation (2.6). This is sufficient to show that the latter scalar field action has the same features as the usual one. However, we would like to see that the equation of motion of the scalar field, φ, in Eq. (2.8) is indeed the Klein-Gordon equation.

The variation with respect to φ yields the following equation:

$$\delta\varphi : \partial_\mu(\sqrt{-g}\pi^\mu) = 0. \tag{2.11}$$

Plugging in the form of the multiplier,

$$\partial_\mu[\sqrt{-g}g^{\mu\nu}(\nabla_\nu\varphi)] = 0. \tag{2.12}$$

We have used the fact $\nabla_\nu\varphi = \partial_\nu\varphi$ for a scalar quantity. This expression can be rewritten in terms of only covariant derivatives by using the definition of the covariant derivative of a tensor density [2]. For a tensor of rank 1:

$$\nabla_\mu[\sqrt{g}T^\mu] = (\partial_\mu\sqrt{g})T^\mu - \Gamma^\rho_{\rho\mu}\sqrt{g}T^\mu + \sqrt{g}\nabla_\mu T^\mu. \tag{2.13}$$

So we have that

$$\begin{aligned}
0 = \delta_\mu[\sqrt{-g}g^{\mu\nu}(\nabla_\nu\varphi)] &= (\delta_\mu\sqrt{-g})g^{\mu\nu}(\nabla_\nu\varphi) + \sqrt{g}[(\delta_\mu g^{\mu\nu})(\nabla_\nu\varphi)] \\
&= +\Gamma^\rho_{\rho\mu}\sqrt{-g}g^{\mu\nu}(\nabla_\nu\varphi) - \Gamma^\rho_{\rho\mu}\sqrt{g}g^{\mu\nu}(\nabla_\nu\varphi) \\
&= [\delta_\mu\sqrt{-g} - \Gamma^\rho_{\rho\mu}\sqrt{-g}]g^{\mu\nu}(\nabla_\nu\varphi) \\
&= +\sqrt{-g}\nabla_\mu[g^{\mu\nu}(\nabla_\nu\varphi)] \\
&= \nabla_\mu[\sqrt{-g}g^{\mu\nu}(\nabla_\nu\varphi)].
\end{aligned}$$

By the metric compatibility of the Levi-Civita connection,

$$\nabla_\mu[\sqrt{-g}g^{\mu\nu}(\nabla_\nu\varphi)] = \sqrt{-g}g^{\mu\nu}\nabla_\mu(\nabla_\nu\varphi) = \sqrt{-g}\,\Box\,\varphi, \tag{2.14}$$

where we used the definition of the d'Alembertian $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$.

Finally, droping the $\sqrt{-g}$, we obtain

$$\Box\,\varphi = 0, \tag{2.15}$$

which is the Klein-Gordon equation.

As we have seen, equations (2.6) and (2.8) are equivalent both on an action level, as well as in the equations of motion they produce.