Chapter 1

Vacuum BF Gravity

Vacuum gravity, in general, is understood as the description of the effects of the gravitational field with not any form of matter coupled to it. In general relativity and newer descriptions of gravity, matter is composed of four types of fields, (i) the electromagnetic field, (ii) Yang-Mills fields, (iii) the scalar field, and (iv) fermions. In this thesis, we are especially interested in coupling the scalar field to a BF gravity formulation.

From a classical point of view, vacuum gravity obeys the Einstein field equations in vacuum [5] [4]:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = 0, \tag{1.1}$$

where $R_{\mu\nu}$ is the Ricci tensor, contracted from the first and third indices of the Riemann curvature tensor, $R^{\rho}{}_{\mu\sigma\nu} \to R^{\rho}{}_{\mu\rho\nu}$, $g_{\mu\nu}$ is the metric tensor, $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar, and Λ is the cosmological constant, which is interpreted as non-zero vacuum energy in field theory. We follow the notation in which Greek indices stand for spacetime indices. The first two terms of the left hand side of the previous equation are referred to together as the Einstein tensor, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$.

If one finds the Einstein tensor to be an unsual mathematical entity, it would be a good idea to state some of its properties to convince ourselves that it is the right choice to describe gravity. $G_{\mu\nu}$ was constructed in order to fulfill the following considerations, that [4]

1. $G_{\mu\nu} = 0$ in flat spacetime,

2. $G_{\mu\nu}$ must be constructed out of the metric and the Riemann tensors,

3. and that $G_{\mu\nu}$ be linear in the Riemann tensor, symetric and a second degree tensor, in order of it to be compatible (in the colloquial sense of the word) with the energy-momentum tensor from field theory, $T_{\mu\nu}$, and have null divergence, $\mathrm{div}(G\mu\nu) = 0$.

A better description of the Einstein field equations can be written in terms of a non-coordinate basis. Its vacuum form is:

$$*\,F_{IJKL} + *F_{IKLJ} + *F_{ILJK} - \Lambda\varepsilon_{IJKL} = 0. \tag{1.2}$$

We call F_{IJ} the curvature of the spin conection. Lorentz indices are represented by Latin letters. While the two descriptions are equivalent, it is important to point out that there exists a fundamental difference in that (1.2) makes no reference to the metric tensor field, but rather to the tetrad field or *vierbein*, e^I. Among other advantages, the tetrad formulation of gravity admits the coupling of fermions, while the previous one fails to do so.

Equations (1.1) and (1.2) can be obtained from variational principles, Therefore with the appropriate choice of some Lagrangian (density) one can make a full description of gravity. Hilbert was the first one to propose an action principle for the equations of motion (1.1). He proposed [5] [4]

$$S_H[g] = \int_{\mathcal{M}^4} \sqrt{-g}R\, d^4x, \tag{1.3}$$

with $g = \det(g_{\mu\nu})$. As it turns out, the simplest Lagrangian density, $\sqrt{-g}R$

yields (1.1) with zero vacuum energy when one performs the variation with respect to the (inverse) metric tensor. (1.3) is known as the Einstein-Hilbert action. Note that by making $R \to R - 2\Lambda$ one obtains exactly (1.1).

Furthermore, one can make the connection independent of the metric tensor, and therefore make the curvature exclusively dependent of the connection. An action principle that allows for this is [5] [4]

$$S_H[g, \Gamma] = \int_{\mathcal{M}^4} \sqrt{-g} g^{\mu\nu} R_{\mu\nu}[\Gamma] \, d^4x. \tag{1.4}$$

This equation is known as the Hilbert-Palatini Action. The variation with respect to the metric tensor yields (1.1), once again, up to the cosmological constant, which can be introduced by adding $-2\sqrt{-g}\Lambda$ to the lagrangian in (1.4), while the variation with respect to the connection gives rise to the torsion free condition,

$$\nabla g_{\mu\nu} = 0, \tag{1.5}$$

which indicates that Γ is the Levi-Civita connection. We call ∇ the covariant derivative as defined by the Levi- Civita connection.

The Palatini formalism is more conveniently expressed in terms of the tetrad field:

$$S[e, A] = \int_{\mathcal{M}^4} \left[*(e^I \wedge e^J) \wedge F_{IJ}[A] - \frac{\Lambda}{12} \varepsilon_{IJKL} e^I \wedge e^J \wedge e^K \wedge e^L \right]. \tag{1.6}$$

A is identified *a posteriori* as the spin connection when one performs the variation of (1.6) with respect to the connection. The variation with respect to the tetrad yields precisely (1.2).

1.1 Introducing BF gravity

A BF theory has the following structure:

$$S[B, A, \psi_i] = \int\limits_{\mathcal{M}^4} \left[B^{IJ} \wedge F_{IJ}[A] + G[B, \psi_i] \right], \tag{1.7}$$

where B^{IJ} is a 2-form which plays the role of the dynamical tensor field, replacing the metric tensor, $F^{IJ}[A]$ represents the curvature of some connection A, and G represents a constraint which depends of $i = 1, \ldots, n$ multipliers ψ_i. The first term in (1.7) is said to be topological, for it conferes no degrees of freedom to the theory. It is by introducing the constraint G to the dynamical field that the theory acquires degrees of freedom, and hence becomes physical.

The first action of the BF type proposed to describe gravity was constructed by Plebański, namely, [6]

$$S_P[\Sigma, A] = \int\limits_{\mathcal{M}^4} \left[\Sigma_i \wedge F^i[A] + \Lambda \Sigma_i \wedge \Sigma^i \right], \tag{1.8}$$

with $i = 1, 2, 3$, and satisfying the constraints

$$3\Sigma^i \wedge \Sigma^j = \delta^{ij} \Sigma_k \wedge \Sigma^k = \delta^{ij} \tilde{\Sigma}_k \wedge \tilde{\Sigma}^k = 0, \tag{1.9}$$

and A^i is a complex self-dual connection in the form of 3 complex 1-forms (as opposed to the spin connection with 6 real 1-forms). S_P has come to be known as the Plebański action, and became the first motivation to explore the possibility of describing gravity with constrained BF theories. On this matter, it can be readily noticed that different constraints G will produce different action principles. Some succesful approaches include actions of the type,

$$S[B, A, \psi, \mu] = \int\limits_{\mathcal{M}^4} \left[B^{IJ} \wedge F_{IJ}[A] + -\frac{1}{2} \psi_{IJKL} B^{IJ} \wedge B^{KL} + \mu H[\psi] \right], \tag{1.10}$$

with μ some Lagrange multiplier. Which can lead to two conditions through the variation of μ, namely,

$$\begin{aligned} H_1 &= \psi_{IJ}{}^{KL} = 0, \\ H_2 &= \varepsilon^{IJKL}\psi_{IJKL} = 0. \end{aligned} \tag{1.11}$$

One can recover the Holst Action Principle [7],

$$S[e,A] = \int\limits_{\mathcal{M}^4} [*(e^I \wedge e^J) + \frac{1}{\gamma} e^I \wedge e^J] \wedge F_{IJ}[A], \tag{1.12}$$

by substituting the equations of motion into the original action (1.10), using the conditions (1.11). Some comments on the Holst action principle will be made at the end of section 1.2. There is however, another natural way to recover (1.12), as we will see in the following section.

1.2 The CMPR action principle

This section is based on the original article [8].

As we have seen, conditions of the form (1.11) have been explored with action principles of the BF type of the form of Eq. (1.10). The CMPR action principle introduces a more general constraint that gives rise to the Holst action in a natural fashion. In terms of six antisymmetric 2-forms, $B^{IJ} = -B^{JI}$; an SO(3,1) connection, $A^I{}_J$ which gives rise to the curvature $F_{IJ}[A] = dA^I{}_J + A^I{}_K \wedge A^K{}_J$; a Lagrange multiplier ψ_{IJKL} which satisfies $\psi_{IJKL} = \psi_{KLIJ}$, $\psi_{IJKL} = -\psi_{JIKL}$, and $\psi_{IJKL} = -\psi_{IJLK}$, leaving 21 independent components; and one Lagrange multiplier μ, the CMPR action principle has the following form:

$$S[B,A,\psi,\mu] = \int\limits_{\mathcal{M}^4} [B^{IJ} \wedge F_{IJ}[A] - \frac{1}{2}\psi_{IJKL} B^{IJ} \wedge B^{KL} \\ - \mu(a_1 \psi_{IJ}{}^{IJ} + a_2 \psi_{IJKL} \varepsilon^{IJKL})]. \tag{1.13}$$

The variation of with respect to the independent fields gives rise to the equations of motion.

To first order, varying the dynamical field, B^{IJ}, yields

$$\delta B : F_{IJ}[A] - \psi_{IJKL} B^{KL} = 0. \tag{1.14}$$

The variation with respect to the connexion yiels

$$\delta A : DB^{IJ} = 0, \tag{1.15}$$

where D is the covariant derivative with respect to the Lorentz connection A. In particular, the covariant derivative of a 2-form is defined as

$$DB^{IJ} \equiv dB^{IJ} + A^I{}_K \wedge B^{KJ} + A^J{}_K \wedge B^{IK}. \tag{1.16}$$

Finally, one can obtain the following equations through the variation of the multipliers:

$$\delta\psi : B^{IJ} \wedge B^{KL} + 2a_1 \mu \eta^{[I|K|} \eta^{J]L} + 2a_2 \mu \varepsilon^{IJKL} = 0, \tag{1.17}$$

and

$$\delta\mu : a_1 \psi_{IJ}{}^{IJ} + a_2 \psi_{IJKL} \varepsilon^{IJKL} = 0. \tag{1.18}$$

In fact, we see that the variation with respect to μ imposes an additional constraint to the Lagrange multiplier ψ_{IJKL}. Also, by solving Eq. (1.17) we know the form of B^{IJ},

$$B^{IJ} = \alpha * (e^I \wedge e^J) + \beta e^I \wedge e^J \tag{1.19}$$

with the coefficients a_2, a_1, α, and β related by

$$\frac{a_2}{a_1} = \frac{\alpha^2 + \sigma\beta^2}{4\alpha\beta}. \tag{1.20}$$

We can now plug in the form of the dynamical field (1.19) into the equation of motion (1.15):

$$\begin{aligned} d[\alpha * (e^I \wedge e^J) + \beta e^I \wedge e^J] + A^I{}_K \wedge [\alpha * (e^I \wedge e^J) + \beta e^I \wedge e^J] \\ + A^J{}_K \wedge [\alpha * (e^I \wedge e^J) + \beta e^I \wedge e^J] = 0. \end{aligned} \tag{1.21}$$

This equation is equivalent to

$$de^I + A^I{}_K \wedge e^K = 0, \tag{1.22}$$

if $\det(e^I_\mu) \neq 0$. This is the exact definition of the spin connection $A = A[e]$. We can verify that this is true by performing a counting argument: The expression (1.21) are 6 equations for 3-forms on a 4-dimentional manifold, yielding a total of $6 \times 4!/(4-3)!3! = 24$ independent equations, and in (1.22) there are 4 equations for 2-forms on a 4-dimentional manifold, yielding $4 \times 4!/(4-2)!2! = 24$ independent equations.

We notice that the second and third term of (1.13) vanish by virtue the equations of motion of ψ and μ, (1.18) and (1.17). Having identified the connection, we can now subsitute for the form of B^{IJ}, Eq. (1.19), into the first and only surviving term of the CMPR action principle (1.13). We obtain

$$S[e, A] = \alpha \int_{\mathcal{M}^4} [{}^*(e^I \wedge e^J) + \frac{\beta}{\alpha} e^I \wedge e^J] \wedge F_{IJ}[A]. \tag{1.23}$$

This is exactly the Holst action principle multiplied by an arbitrary constant α. We identify the Immirzi parameter as α/β.

As claimed, one can fully recover the Holst action purely from the CMPR action principle. For its importance, we wish to discuss on the importance of this action. It was proposed in 1996 by Sören Holst as a generalization of the Hilbert-Palatini action principle in order to derive Barbero's Hamiltonian, which gives rise to the canonical variables of the phase space of general relativity, from a variational principle.

Ashtekar proposed a pair of geometrical variables for general relativity which led to a simple Hamiltonian. These, known as Ashtekar variables, are succesful in that they allow the use of loop variables at classical and quantum level. However, a difficuly arose in that the variables must be complex in order to describe Lorentzian spacetime, therefore asking for additional reality conditions to be imposed. Later, Barbero proposed a new set of variables, known as Ashtekar-Barbero variables, which lead to a real formulation of general relativity, without the need to impose additional constraints [9] - although the simplicity of Ashtekar's Hamiltonian is lost.

The variational principle from which Barbero's formulation is derived includes a non-zero parameter, known as the Immirzi parameter. It reads as follows [7]:

$$S[e,A] = \int\limits_{\mathcal{M}^4} [*(e^I \wedge e^J) + \frac{1}{\gamma} e^I \wedge e^J] \wedge F_{IJ}[A]. \tag{1.24}$$

The Immirzi parameter appears as γ in the so-called Holst term in the previous equation. One can see that the structure of the Holst action is a natural extension of the Hilbert-Palatini action. As a matter of fact, by making $1/\gamma = 0$ one can recover the latter.

Another interesting case consists in making $1/\gamma = \imath$, the imaginary unit. This is equivalent to writing down a Hilbert-Palatini Lagrangian in which the curvature F_{IJ} is replaced by its self-dual form. This would lead to Ashtekar's Hamiltonian. The most interesting case is when one sets γ to be a real number. This gives rise to a plethora of quantum theories of gravity. As it turns out, the parameter is typically normalized to match the Berkenstein-Hawking entropy of a black hole.

A fundamental feature of the Holst action is that the Holst term is exactly equal to zero from a classical point of view, making the action equivalent to Hilbert-Palatini. This property is guaranteed by the curvature which has to obey the Bianchi identities, once the spin connection has been idetified, namely, it satifies the following equation:

$$F_{IJKL} + F_{IKLJ} + F_{ILJK} = 0. \tag{1.25}$$

By virtue of this identity, the Holst term in (1.24) is:

$$\begin{aligned}
\int\limits_{\mathcal{M}^4} e^I \wedge e^J \wedge F_{IJ}[A] &= \int\limits_{\mathcal{M}^4} e^I_\mu e^J_\nu F_{IJ\alpha\beta} \varepsilon^{\alpha\beta\mu\nu} d^4x \\
&= \sigma\varepsilon \int\limits_{\mathcal{M}^4} \det(e^I_\mu) e^I_\mu e^J_\nu e^\alpha_K e^\beta_L e^\mu_M e^\nu_N F_{IJ\alpha\beta} \varepsilon^{KLMN} d^4x \\
&= \sigma\varepsilon \int\limits_{\mathcal{M}^4} \det(e^I_\mu) \delta^I_M \delta^J_N e^\alpha_K e^\beta_L F_{IJ\alpha\beta} \varepsilon^{KLMN} d^4x \\
&= \sigma\varepsilon \int\limits_{\mathcal{M}^4} \det(e^I_\mu) F_{IJKL} \varepsilon^{IJKL} d^4x.
\end{aligned}$$

So now we can verify that $F_{IJKL}\epsilon^{IJKL} = 0$,

$$\begin{aligned}
F_{IJKL}\varepsilon^{IJKL} &= F_{0123}\varepsilon^{0123} + F_{0132}\varepsilon^{0132} + F_{0213}\varepsilon^{0213} + F_{0231}\epsilon^{0231} \\
&+ F_{0312}\varepsilon^{0312} + F_{0321}\varepsilon^{0321} + F_{1023}\varepsilon^{1023} + F_{1032}\varepsilon^{1032} \\
&+ F_{1203}\varepsilon^{1203} + F_{1230}\varepsilon^{1230} + F_{1302}\varepsilon^{1302} + F_{1320}\varepsilon^{1320} \\
&+ F_{2013}\varepsilon^{2013} + F_{2031}\varepsilon^{2031} + F_{2103}\varepsilon^{2103} + F_{2130}\varepsilon^{2130} \\
&+ F_{2310}\varepsilon^{2310} + F_{2301}\varepsilon^{2301} + F_{3012}\varepsilon^{3012} + F_{3021}\varepsilon^{3021} \\
&+ F_{3102}\varepsilon^{3102} + F_{3120}\varepsilon^{3120} + F_{3201}\varepsilon^{3201} + F_{3210}\varepsilon^{3210} \\
&= 4[(F_{0123} + F_{0231} + F_{0312}) - (F_{0213} + F_{0132} + F_{0321})] \\
&= 0.
\end{aligned}$$

In the last line we used the identity (1.25). We have adhered to the convention that the Levi- Civita symbol is equal to $\varepsilon = 1$ for even permutations.

Since the Holst term provides no degrees of freedom, it is said to be a topological term. Indeed,

$$S[e, A] = \int_{\mathcal{M}^4} e^I \wedge e^J \wedge F_{IJ}[A] \tag{1.26}$$

has long been conjectured to be a topological theory. Recently, Liu, Montesinos, and Perez have shown that this conjecture is correct in the absence of boundaries and they argued that the quantization of the theory might be relevant in the study of the entropy of black holes in loop quantum gravity [10].

A closing comment on the Holst action principle would be to point out its importance for the non-perturvative quantization of the gravitational field, especially from the loop quantum gravity viewpoint. As a matter of fact, it plays a central role in said theory, being the starting point of the canonical and path integral quantization of the gravitational field.

1.3 The Montesinos-Velázquez action principle

This section is based on [11].

The Einstein field equations including a non-zero vacuum enery term in the form of the cosmological constant, as we have seen in Eq. (1.2), are

$$^*F_{IJKL} + {^*F_{IKLJ}} + {^*F_{ILJK}} = \Lambda \varepsilon_{IJKL}. \tag{1.27}$$

As in the case in which the cosmological constant does not appear, it is possible to build an action that will yield Eq. (1.27). In terms of the same fields that were introduced for CMPR, along with three constants $\mathcal{H}$, l_1, and l_2, and the cosmological constant Λ , the following action principle is proposed:

$$S[B,A,\psi,\mu] = \int\limits_{\mathcal{M}^4} [B^{IJ} \wedge F_{IJ}[A] - \frac{1}{2}\psi_{IJKL}B^{IJ} \wedge B^{KL} - \mu(a_1 {\psi_{IJ}}^{IJ} + a_2 \psi_{IJKL}\varepsilon^{IJKL} - \mathcal{H}) + l_1 B_{IJ} \wedge B^{IJ} + l_2 B_{IJ} \wedge *B^{IJ}]. \tag{1.28}$$

The modification includes some constant $\mathcal{H}$ that will give rise to a new relationship between the invariants ${\psi_{IJ}}^{IJ}$ and $\varepsilon^{IJKL}\psi_{IJKL}$, as well as two natural additions l_1 and l_2 which appear on the volume terms $B_{IJ} \wedge B^{IJ}$ and $B_{IJ} \wedge *B^{IJ}$.

The variation with respect to the connection A and the multiplier ψ_{IJKL} yield the old equations of motion (1.15) and (1.17), namely:

$$\delta A : DB^{IJ} = 0, \tag{1.29}$$

and

$$\delta\psi : B^{IJ} \wedge B^{KL} + 2a_1\mu\eta^{[I|K|}\eta^{J]L} + 2a_2\mu\varepsilon^{IJKL} = 0. \tag{1.30}$$

The fact that these two equations hold allows for the form of B^{IJ} to remain the same, in virtue of Eq. (1.30),

$$B^{IJ} = \alpha * (e^I \wedge e^J) + \beta(e^I \wedge e^J), \tag{1.31}$$

along with the coefficients relationship

$$\frac{a_2}{a_1} = \frac{\alpha^2 + \sigma\beta^2}{4\alpha\beta}; \tag{1.32}$$

but also protects the connection, in virtue of Eq. (1.29), letting A remain the spin connection.

We can convince ourselves of the claim made, that the expressions $B_{IJ} \wedge$

B^{IJ} and $B_{IJ} \wedge *B^{IJ}$ are the volume elements of the theory by substituting for the tetrad fields. We find:

$$\begin{aligned} B_{IJ} \wedge B^{IJ} &= \alpha\beta\varepsilon_{IJKL} e^I \wedge e^J \wedge e^K \wedge e^L, \text{and} \\ B_{IJ} \wedge *B^{IJ} &= \frac{1}{2}\left(\alpha^2\sigma + \beta^2\right)\varepsilon_{IJKL} e^I \wedge e^J \wedge e^K \wedge e^L. \end{aligned} \tag{1.33}$$

On the other hand Eqs. (1.14) and (1.18) no longer remain valid. Instead, performing the variation with respect to the dynamical field, B^{IJ}, yields

$$\delta B : F_{IJ}[A] - \psi_{IJKL} B^{KL} + 2l_1 B_{IJ} + 2l_2 {}^* B_{IJ} = 0, \tag{1.34}$$

and the variation with respect to the Lagrange multiplier, μ, results in

$$\delta\mu : a_1 \psi_{IJ}{}^{IJ} + a_2 \psi_{IJKL} \varepsilon^{IJKL} - \mathcal{H} = 0. \tag{1.35}$$

We see that Eq. (1.34) establishes a new relationship between the curvature and the dynamical field, implying therefore a new relationship between the curvature and the tetrad field. Equation (1.18), as we said before, imposes a new relationship between the invariants.

As in the previous case, we rewrite the action (1.28) in terms of the tetrad field by substituting for the 2-forms in Eq. (1.31):

$$\begin{aligned} S[e, A] = \int_{\mathcal{M}^4} & [[\alpha^*(e^I \wedge e^J) + \beta e^I \wedge e^J] \wedge F_{IJ}[A] + \mu\mathcal{H} \\ & + [l_1\alpha\beta + \frac{l_2}{2}(\alpha^2\sigma + \beta^2)]\varepsilon_{IJKL} e^I \wedge e^J \wedge e^K \wedge e^L]. \end{aligned} \tag{1.36}$$

We are left to verify that the BF theory leads to natural constraints for the curvature that will guarantee that the Bianchi identities (1.25) hold, as the spin connection obliges that they be satisfied. These in turn will establish the relationship between the new constants, $\mathcal{H}$, l_1, and l_2, and the cosmological constant, Λ, that will allow for Eq. (1.36) to be exacly equivalent to the Holst action with non-zero vacuum energy.

Writing equation (1.34) in terms of the tetrad field, and the expanded form of the curvature $F_{IJ} = \frac{1}{2}F_{IJKL}e^K \wedge e^L$ has the following structure

$$\begin{aligned} F_{IJKL}e^K \wedge e^L = 2[\alpha\psi^*{}_{IJKL} + \beta\psi_{IJKL} - (l_1\alpha + l_2\beta)\varepsilon_{IJKL} \\ - (l_1\beta + l_2\alpha\sigma)(\eta_{IK}\eta_{JL} - \eta_{JK}\eta_{IL})]e^K \wedge e^L. \end{aligned} \tag{1.37}$$

Where we have made use of the definition of the right dual $\psi^*{}_{IJKL} = \frac{1}{2}\varepsilon_{KL}{}^{MN}\psi_{IJMN}$ in the previous expression. The Bianchi identities impose the following restrictions on ψ_{IJKL}:,

$$\begin{aligned} 2\alpha(\psi^*{}_{IJKL} + \psi^*{}_{IKLJ} + \psi^*{}_{ILJK}) + 2\beta(\psi_{IJKL} + \psi_{IKLJ} + \psi_{ILJK}) \\ -6(l_1\alpha + l_2\beta)\varepsilon_{IJKL} = 0. \end{aligned} \tag{1.38}$$

This restriction allows us to remove to rewrite the curvature in such a way that it naturally satisfies the Bianchi identities, once the relationship between the constants has been determined, by directly substituting for the term $-2(l_1\alpha + l_2\beta)\varepsilon_{IJKL}$ in Eq. (1.38) into Eq. (1.37):

$$\begin{aligned} F_{IJKL} = 2\alpha\psi^*{}_{IJKL} + 2\beta\psi_{IJKL} - \frac{2}{3}\alpha(\psi^*{}_{IJKL} + \psi^*{}_{IKLJ} + \psi^*{}_{ILJK}) \\ - \frac{2}{3}\beta(\psi_{IJKL} + \psi_{IKLJ} + \psi_{ILJK}) - 2(l_1\beta + l_2\alpha\sigma)(\eta_{IK}\eta_{JL} - \eta_{JK}\eta_{IL}). \end{aligned} \tag{1.39}$$

As we said, the Bianchi identities are the key to relating the invariants. The contraction of Eq. (1.38) with the Levi-Civita tensor density ε_{IJKL} gives rise to a relationship between the invariants $\psi_{IJ}{}^{IJ}$ and $\varepsilon^{IJKL}\psi_{IJKL}$,

$$\alpha\sigma\psi_{IJ}{}^{IJ} + \beta^*\psi_{IJ}{}^{IJ} - 12\sigma(l_1\alpha + l_2\beta) = 0, \tag{1.40}$$

while the variation with respect to μ in equation (1.35) also relates them. Combining these two equations, together with (1.32) one obtains

$$\beta\psi_{IJ}{}^{IJ} + \alpha\psi^*{}_{IJ}{}^{IJ} + 12\beta l_1 + 12\frac{\beta^2}{\alpha}{}_2 - \frac{2\beta}{a_1}\mathcal{H} = 0. \tag{1.41}$$

Contracting $F_{IJKL} \to F^{IJ}{}_{IJ}$ as it is expressed in Eq. (1.37) and combining it with the contraction of the Einstein field equations (1.27) with the Levi-Civita symbol, $4\Lambda = F^{IJ}{}_{IJ}$, one obtains

$$\alpha\psi^*{}_{IJ}{}^{IJ} + \beta\psi_{IJ}{}^{IJ} = 12(l_1\beta + l_2\alpha\sigma) + 2\Lambda. \tag{1.42}$$

The last equation is the only exception for the Bianchi identities to be satisfied. This can be fixed by determining the correct relation between the consants $\mathcal{H}$, l_1, l_2, and Λ together with a_1 and a_2. One can obtain such relationship by combining (1.41) and (1.42) into

$$\mathcal{H} = a_1\left[12l_1 + 4!\sigma\frac{a_2}{a_1}l_2 + \frac{\Lambda}{\beta}\right]. \tag{1.43}$$

This choice of $\mathcal{H}$ guarantees that all of the Bianchi identities will be satisfied. Additionally, by plugging this result into (1.36) one can readily recover

$$S[e,A] = \alpha\int\limits_{\mathcal{M}^4}\left[[{}^*(e^I\wedge e^J) + \frac{\beta}{\alpha}e^I\wedge e^J]\wedge F_{IJ}[A] - \frac{\Lambda}{12}\varepsilon_{IJKL}e^I\wedge e^J\wedge e^K\wedge e^L\right], \tag{1.44}$$

which is the Holst action principle with cosmological constant, as expected.

A closing comment on this chapter would be to summarize the importance of BF theories. First of all, we have seen that the study of said theories gives rise to a rich approach to the study of gravity since, by handling the restrictions that one imposes on the dynamical field, one can obtain gravitational theories that fulfill different purposes. Additionally, we have explored in detail two of such BF theories that correctly describe gravity with Immirzi parameter with and without cosmological constant, by correctly handling the equations of motion. This is a display of the power of the BF formulation of gravity. Finally and most importantly, we argued that a major motivation for the study of BF gravity is that it allows for the projects of quantization of the gravitational field to continue from its canonical quantization and path integral viewpoints.