Chapter 3

Optical systems and Imaging

> "Fourier's Theorem is not only one of the most beautiful results of modern analysis, but it is said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics."
>
> *Lord Kelvin*

The material presented in this chapter is useful to model the optics of the eye. We will start by discussing imaging optical systems (OS) in general, their parts and how to model and study them through Fourier Theory.

3.1 Optical systems and image formation

A system capable of being used for imaging is called an imaging system. The main elements from which optical systems are built are: thin lenses, compound lenses, prisms or plane mirrors, spherical or parabolic mirrors, and aperture stops. Each one of them accomplish different tasks. Thin lenses converge or diverge bundles of light rays, for example in spectacles or magnifying glasses. Compound lenses are designed to correct various aberrations. Monochromatic aberrations are corrected by lenses, but chromatic aberrations are corrected by doublets.

On the one hand, plane mirrors or prisms alter the direction of the optical path and may be used to invert an image. On the other hand, spherical or parabolic mirrors, in some cases, can replace lenses. One example of its use is in large newtonian telescopes.

3.1.1 Pupils in imaging

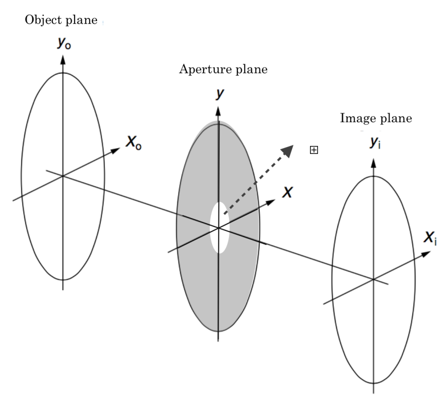

Figure 3.1: The standard coordinate system that will be used for all mathematical analyses throughout this text.

Every optical system has some component that limits the light cone that is accepted from an axial object. Aperture stops limit the diameter which determines the amount of light that reaches the image plane, as shown in the aperture plane on Fig 3.1. A wider aperture enlarges the aforementioned cone of light that impacts on the Image plane.

These stops are also called pupils. In every imaging system we can identify the entrance pupil and the exit pupil.

The entrance pupil is an image of the aperture stop as seen from the object side. Defines the cone of light accepted by the optical system. The importance of the entrance pupil is that the brightness of the image depends on this cone angle. The larger the acceptance angle, the more light that is collected from each object point, and hence the brighter the image. The exit pupil is the image of the aperture stop, as seen from the image side of the optic. It defines the cone angle of light converging to the image point. This is important in determining the image resolution that is set by diffraction. Entrance and exit Pupils are images of each other.

In a nutshell, the pupils define the amount of light accepted by and emitted from the optical system, and the entrance and exit pupil are the images of the stop from different sides. So, the entrance and exit pupils must also be images of each other.

The complex pupil function

As we've mentioned before the complex pupil function $P(x, y)$ can describe the effect the optical system of the eye has on the light it receives.

$$P(x, y) = A(x, y)e^{-ikW(x,y)} \tag{3.1}$$

A(x,y) is the amplitude component of the function, and it defines the size, shape, and transmission of the optical system. $W(x, y)$ represents the aberration, a topic which will be discussed later. In this case we are going to define our amplitude component as a circ function (1.10) that defines a circular aperture,

$$A(x, y) = circ(r) = \begin{cases} 1; & if \quad r \quad \leq \quad 1 \\ 0; & elsewhere \end{cases} \tag{3.2}$$

3.1.2 Describing an Imaging System

Most imaging systems, including the eye, are designed to take points of light out in the world and relay them to a perfect focus on an image plane as in Fig. 3.2

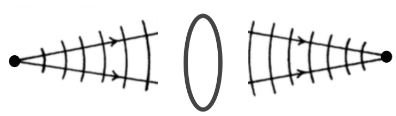

Figure 3.2: Expanding wavefronts from a point of light are captured by an imaging system and converted to converging spherical wavefronts. These converging wavefronts focus to a point.

The image illustrates a simple imaging system composed by a single lens. A point of light produces a series of expanding spherical wavefronts. These wavefronts propagate outwards and are intercepted by the lens. The lens is designed to convert the expanding wavefronts into converging perfect spherical wavefronts that focus to a point. If the distances from the object to the lens and from the lens to the image are s_0 and s_1 respectively, for a lens of negligible thickness, in air, the distances are related by the Thin Lens Formula. Thus described in Eq. 3.1

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f} \tag{3.3}$$

If the point sources are far from the lens $z \to \infty$, the wavefronts interacting with the lens will expand to such a degree that they will become perfectly flat plane waves.

In terms of the eye, the ideal performance is that plane waves from a distant point are converted to perfectly spherical waves that converge to a precise point on the retina, where cone cells are densely packed. For points closer to the eye, ideally the eye "accommodates" to convert the diverging wavefronts from the near object into perfectly spherical wavefronts that focus to a point on the retina.

In a nutshell, if we know how the wave front of a parallel beam is altered in a given direction, then we can predict how the image will be formed (Fig.1.2).

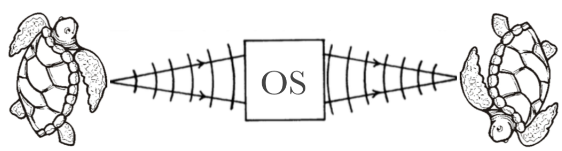

Figure 3.3: "Black box" Optical system. In the eye's OS, the image reconstructed from the exit pupil appears horizontally flipped.

Aberrations occur when the wavefront coming through the eye differs from the ideal one, this results on blurry vision caused by an imperfect image being reconstructed in the retina . We can see an example in Fig.3.4, but the concept will be covered with more detail in the following chapter.

A very common optical aberration is called myopia, Fig. 3.5 is a representation of how a person that suffers from myopia perceives the world. Sharp at a close range but blurry and undefined after a certain distance.

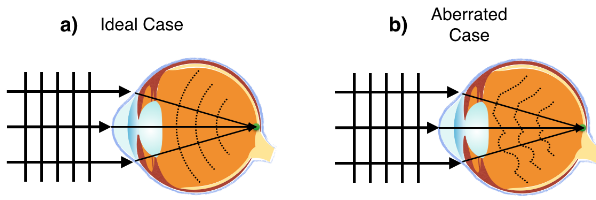

Figure 3.4: Example of wavefront aberration in the eye.

Figure 3.5: This is a visual approximation of how a person with myopia sees. Image taken from WebMD (Brian S. Boxer Wachler).

3.1.3 The Wave Equation

Diffraction is a phenomenon that occurs for any kind of wave, and refers to the propagation of a wave as it goes past an obstacle or through an aperture.

When we ignore polarization, it is sufficient in optics to model light by using the *scalar wave equation* (1.1).

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t^2} \tag{3.4}$$

When $\psi = \vec{E}$ and

$$E\left(\vec{r}, t\right) = E_0 e^{ikr} e^{-iwt}$$

the scalar wave equation becomes the Hemholtz Equation.

$$\nabla^2 \psi = \mu_0 \epsilon_0 \frac{\partial \psi}{\partial t^2} \tag{3.5}$$

A solution to the Helmoltz Equation is

$$\psi_{diff}(\vec{R}) = \frac{A}{i\lambda} \int P(x,y) \frac{e^{ikd}}{d} dxdy \tag{3.6}$$

where A is a constant and $d = z_0 - z$ represents the distance between the aperture plane and the diffraction plane. The integral is calculated over the plane of the aperture, and $P(x,y)$ is the fraction of the field that transmits through the aperture, the *pupil function*, which will be discussed further at a later occasion.

Note that $P(x,y) = 0$ for any points x and y that are blocked by the aperture.

The distance D is the distance between the point (x,y,z) in the aperture plane, and the point in the plane where we are calculating and observing the diffraction pattern (x_i, y_i, z_i).

Figure 1.2 shows the setup for the integral. Each point in the aperture plane acts like a new source of waves propagating out to the image plane, where the virtual image is formed. The integral sums up all the contributions to get the electric field at a position in the image plane (x_i, y_i, z_i).

3.1.4 The Fresnel Approximation

The solution to the Hemholtz Eq. given in Eq.(1.3) can undergo additional simplification as follows:

First, it is assumed that

$$D \ll d \quad and \quad \frac{1}{d} \approx \frac{1}{D}$$

As a next step

$$d = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

In the *Fresnel approximation* we take

$$\left| \frac{(x-x_o)^2 + (y-y_0)^2}{(z-z_0)^2} \right| \ll 1$$

so that we can expand

$$d = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

$$= D\sqrt{1 + \frac{(x_0 - x)^2}{D^2} + \frac{(y_0 - y)^2}{D^2}}$$

We can use the Taylor approximation to obtain

$$\approx D\left(1 + \frac{1}{2}\frac{(-x_0 - x)^2}{D^2}\right) \tag{3.7}$$

and

$$d = D + \frac{(x_0^2 + {y_0}^2)}{2D} + \frac{(2xx_0 - 2yy_0 + x^2 + y^2)}{2D} \tag{3.8}$$

The first two terms in the approximation (1.4) do not depend on the position in the aperture plane, so they are constants in the integral.

The third term is the part that contains all the interference effects from the different Huygens waves propagating out from the aperture. The full integral becomes

$$\psi_{diff}^{Fresnel}(\vec{R}) = \frac{1}{i\lambda D} e^{ik\frac{x_0^2+y_0^2}{D}} \int P(x,y) e^{ik\frac{-2xx_0-2yy_0+x^2+y^2}{D}} dxdy \tag{3.9}$$

which is known as the *Fresnel Integral*.

3.1.5 The Fraunhofer approximation

The Fraunhofer approximation involves the additional assumption that we can ignore the exponentiated quadratic terms with $e^{\frac{ikx^2}{D}}$ and $e^{\frac{iky^2}{D}}$ in Eq.(1.5)

Then approximate the ikD exponent as

$$kD \approx kD + \frac{k(x_0^2 + y_0^2)}{2D} + \frac{k(-2xx_0 - 2yy_0)}{2D} \tag{3.10}$$

However, the Fraunhofer diffraction pattern only occurs when viewed in the far-field

$$z = \infty$$

As a result, the Fraunhofer diffraction integral becomes

$$\psi_{diff}^{Fraun}(\vec{R}) = \frac{1}{i\lambda D} e^{ikD} e^{ik\frac{x_0^2+y_0^2}{2D}} \int P(x,y) e^{ik\frac{-2xx_0-2yy_0}{2D}} dxdy \tag{3.11}$$

As we can see Fraunhofer's Integral is mathematically identical to the Fourier Transform. What I end up seeing as the intensity can be normalized to see a unitary pattern.

3.1.6 The Point Spread Function

The PSF is the image of a point source formed by the optical system, it can be computed with a FT by using the Fraunhofer integral (for PSFs near the image plane), and describes the response of an imaging system to a point source or point object.

$$PSF\left(x_i, y_i\right) = K \Big| FT\left\{P\left(x, y\right) e^{-ikW(x,y)}\right\} \Big|^2 \tag{3.12}$$

where FT represents the Fourier transform operator and K is a constant. The actual Fourier transform is not discussed in this text. Once the concepts are understood, computations like the Fourier transform can be done numerically by using one of many common software packages such as Matlab (Roorda).

The diffraction theory of PSFs was first studied by Airy in the nineteenth century. He developed an expression for the point-spread function amplitude and intensity of a perfect instrument, free of aberrations (the so-called Airy disc). That is, the Airy disk represents the best focused spot of light that a perfect lens with a circular aperture can make, limited by the diffraction of light. The degree of spreading (blurring) of the point object is a measure for the quality of an imaging system.

The theory of aberrated PSFs close to the optimal focal plane was studied by the Dutch physicists Frits Zernike and Nijboer in the 1930–40s. A central role in their analysis is played by Zernike's circle polynomials that allow an efficient representation of the aberrations of any optical system with rotational symmetry.

The intensity of the Fraunhofer diffraction pattern of a circular aperture (the Airy pattern) is given by the squared modulus of the Fourier transform of the circular aperture

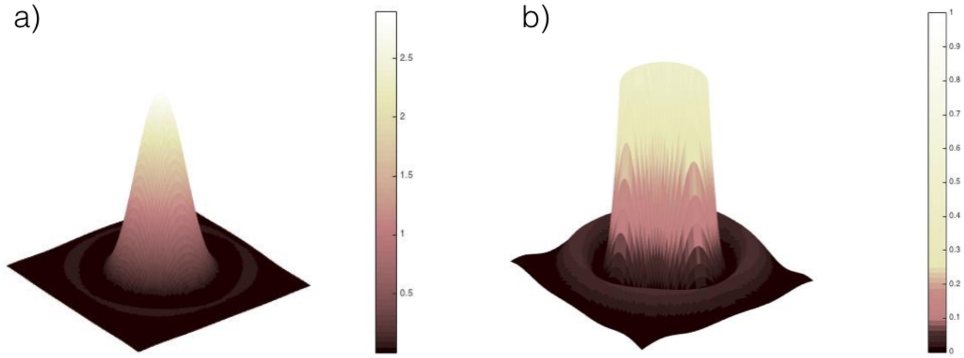

Figure 3.6: Example of an Airy Disk. a) full scale b) detail of the outer minimum rings.

3.2 The lens as a Fourier Transform

The connection between the Fraunhofer diffraction integral and the spatial Fourier transform of the electric field is very useful for basic and applied research. There is an entire sub-field of optics called "Fourier Optics" dedicated to exploiting the relationship between Fourier theory and optics to do interesting science.

As we now, the far-field diffraction pattern can only be observed at infinity. By placing a lens after the diffracting aperture the plane at infinity is imaged onto the focal plane of the lens. This explains why a lens can perform a Fourier transform, the Fraunhofer diffraction pattern at a screen at infinity is the FT of the aperture function of an opening.

The Fraunhofer diffraction integral given in Eq. (1.8) follows the definition of the Fourier Transform (FT) given below

$$F[u'] = A\int_{-\infty}^{\infty} f(x)e^{-i2\pi u'x}dx \tag{3.13}$$

Where we can interpret x as the position in the diffraction plane, and u' as the spatial frequency of the diffraction. Let's remember that the Fraunhofer diffraction pattern only occurs when $z = \infty$

These are the three optical methods of producing two-dimensional Fourier Transforms according to Nussbaum & Phillips (1976). The complete proofs can be found at (Nussbaum and Philips).

1. The Fraunhofer diffraction pattern at a screen at infinity is the FT of the aperture function of an opening
2. The Fraunhofer diffraction pattern at the focal distance f' from a lenses the FT of its pupil function of an opening
3. An object at f is converted into its FT at f' by a lens

The integral method of computing diffraction patterns works, but the integrals can take a long time to compute. We can take advantage of the connection between the Fourier Transform and Fraunhofer diffraction and use a very efficient method of calculating Fourier Transforms called the Fast Fourier Transform (FFT).(Roorda Hecht Goodman)

3.2.1 Defining Fourier's kernel

There are many ways of defining the Fourier Transform, all of them with the same properties, but choosing the right kernel to use is of great importance to better describe and facilitate the understanding of our optic model.

We start with the equation for a diverging wave $u(\xi,\eta) = Ae^{-i\frac{k}{2z}\left[\xi^2+\eta^2\right]}$. Then add a converging lens, $u(\xi,\eta) = P_r(\xi,\eta)\, e^{-i\frac{k}{2f}\left[\xi^2+\eta^2\right]}$. If we remember the Fresnel Integral,

$$u_z(x,y) = Ae^{ikz} \iint u(\xi,\eta) e^{-i\frac{k}{2z}\left[(x-\xi)^2+(y-\eta)^2\right]} d\xi d\eta, \tag{3.14}$$

We define $Ae^{ikz} = A(z)$ for simplicity, and we rewrite the integral in the form

$$u_z(x,y) = A(z)\, e^{-i\frac{k}{2z}r^2} \iint u(\xi,\eta) e^{-i\frac{k}{2z}\rho^2} e^{+i\frac{k}{2z}(2x\xi+2y\eta)} d\xi d\eta. \tag{3.15}$$

So, if $u(\xi,\eta)$ is a lens illuminated by a diverging wave, then

$$u(\xi,\eta) = Ae^{-i\frac{k}{2s_0}\left(\xi^2+\eta^2\right)} P_r(\xi,\eta)\, e^{i\frac{k}{2f}\left(\xi^2+\eta^2\right)} \tag{3.16}$$

Where $e^{-i\frac{k}{2s_0}\left(\xi^2+\eta^2\right)}$ is the center at s_0, and $e^{i\frac{k}{2f}\left(\xi^2+\eta^2\right)}$ the converging lens phase.

Then the integral becomes

$$u\left(x,y\right)=B\left(z\right)\iint u\left(\xi,\eta\right)e^{-i\frac{k}{2z}\rho^{2}}e^{i2\pi(u\xi+v\eta)}d\xi d\eta, \tag{3.17}$$

where $u=\frac{x}{\lambda z}$, and $v=\frac{y}{\lambda z}$. Resulting in

$$u_z\left(u,v\right)=B\left(z\right)\iint Ae^{\frac{-ik}{2s_0}\rho^2}P\left(\rho\right)e^{\frac{-ik}{2f}\rho^2}d\xi d\eta$$

$$u_z\left(u,v\right)=B\left(z\right)\iint Ae^{\frac{-ik}{2z}\rho^2}P\left(\rho\right)e^{i2\pi(u\xi+v\eta)}d\xi d\eta$$

if $z=s_i$ then,

$$-\frac{1}{s_0}-\frac{1}{s_i}+\frac{1}{f}=0$$

because of the Lensmaker Equation Eq.3.18.

$$\frac{1}{s_0}+\frac{1}{s_i}=\frac{1}{f} \tag{3.18}$$

We can finally obtain

$$u_z\left(u,v\right)=B\left(z\right)\iint P\left(\rho\right)e^{i2\pi(u\xi+v\eta)}d\xi d\eta \tag{3.19}$$

Which is the FT of $P(\rho)$, thus the importance of defining the Fourier kernel with 2π in front. A more detailed explanation of this process is given in Chapter 5 of Goodman's Introduction To Fourier Optics.

3.2.2 Wave Aberrations

All optical systems, including the eye, suffer from aberrations. The complex pupil function contains the wave aberration $W(x,y)$. Aberrations are errors introduced by the optical system that cause the perfectly spherical converging wavefronts to distort from their ideal shape, ultimately causing an imperfect focus.

The wavefront is a connected set of points in space which have the same phase. Wave aberration is defined as the difference between the reference wave front, which in most cases is the ideal diffraction-limited wavefront, and the actual wavefront.

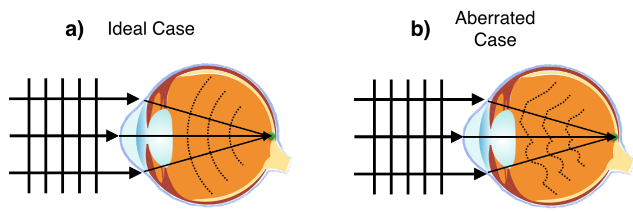

Figure 3.7: To the left: Plane wavefronts incident on the eye are converted to perfectly spherical wavefronts and converge to a point on the retina in the aberration-free case. To the right: In the presence of aberrations, the converging wavefronts are no longer spherical and a blurry spot is formed on the retina

Examples of aberrations in the human eye

Basic aberrations, or low-order aberrations, include a spherical refractive error, also known as defocus in the wavefront world, and astigmatism. These aberrations of the eye cause the wavefronts within the eye to deviate from their ideal shape. In the case of myopia, the converging wavefronts focus too quickly, converging to a point in front of the retina. In the case of hyperopia, the wavefronts do not converge rapidly enough, and the focus ends up behind the retina. In the case of astigmatism, the focus of the wavefront depends on meridian.

The concepts we have learned in this chapter will help us understand how cones are detected and counted. They will also be advantageous to improve current expensive methods, as the techniques presented in this chapter will be used to present a new method for detecting cones in the retina. This method can be implemented for a fraction of the price of high-end systems, and provide more processing speed, as well as more accurate readings.

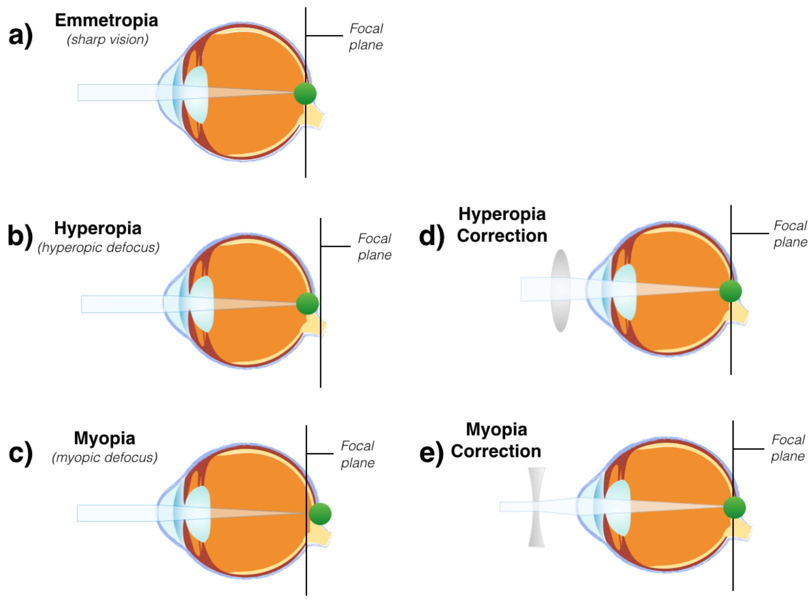

Figure 3.8: Examples of common and easily correctable aberrations in the human eye.