CHAPTER 3. THE RENORMALIZATION GROUP

We now move on to a tool of study that specializes in dealing with the representation of a system as a function of its degrees of freedom. This would serve to our search for different representations of a same model –arguably some of them easier to solve than others- as the methodology set by the renormalization group makes it possible to pass from one to another by means of a coarse graining. As degrees of freedom are usually associated to a scale (of energy for instance), we are more interested on those scales which show some intrinsic interesting features –their characteristic building blocks-, and the effective theory of the system that can derived from them; in this chapter we will review some examples where the renormalization group offers this in an exact manner.

The renormalization group is a mathematical tool that relates changes in a physical system as a function of the scale in which it is studied. It is useful to make calculations in systems with a large number of degrees of freedom, that is, a system defined according to a set $S(s,\mu)$ composed of $s=\{s_1,\dots,s_N\}$ degrees of freedom and $\mu=\{\mu_1,\dots,\mu_m\}$ parameters that can determine a vector in the space of parameters M. A function $H(S)$ describes its dynamics (a partition function, Hamiltonian, etc) and contains the information about the physics of the system. The coarse graining transformation $\mathcal{R}$ maps $\mathcal{R}: S \longrightarrow \tilde{S}$, from the original system to a new $\tilde{S}(\tilde{s},\tilde{\mu})$ with a lower number of degrees of freedom $|\tilde{s}|<|s|$, and with a new vector in the space of parameters M' which may be different from M; this overall results in an effective dynamic function $\widetilde{H}(\tilde{S})$. Hence such coarse graining simplifies the description of the system when N is originally big and, more importantly, if

the functional form of S is preserved, allows to obtain essentially the same system under a different set of parameters. Iterations of this coarse graining mean a mapping into a point as part of a trajectory in $M' = M$. As we are able to solve a system in a *renormalized* model (due to the reduction in the degrees of freedom), the use of this solution to give insight on the initial problem will depend on whether $\widetilde{H}(\tilde{S})$ remains invariant except for $\tilde{\mu}$ in the best of cases and as an approximation otherwise.

3.1 An uncoupled system

We introduce a brief example of a classical free gas to hint at the subtlety of a selective 'coarse graining' parameter dependence implied in constructing different representations of a system vis-à-vis the renormalization group. To obtain the partition function ζ of a single particle we sum all energy states over the phase space of possible positions and momenta:

$$\zeta = \frac{1}{h^3} \int \mathrm{e}^{-\beta \frac{p^2}{2m}} dx^3 dp^3 \tag{39}$$

The energy contribution comes only from the kinetic energy of particle; h, the Planck constant, serves as a normalization factor considering that for each degree of freedom $\vec{x}$ and $\vec{p}$, the correspondent state in the phase space is defined at most by the span of $\Delta x \Delta p = h$[31]; this also allows a correspondence with the quantum-mechanical description.

For a classical system of free, N non-interacting particles, the partition function is the sum over the j-th states α_j of the sum over the i-th components' energies $\epsilon_i(\alpha_j)$:

$$Z = \sum_j \mathrm{e}^{-\beta E_j}$$

$$= \sum_j \exp\left(-\beta \sum_{i=1}^{N} \epsilon_i(\alpha_j)\right)$$

$$= \sum_j \prod_{i=1}^{N} \exp\left(-\beta \epsilon_i(\alpha_j)\right)$$

$$= \prod_{i=1}^{N} \sum_j \exp\left(-\beta \epsilon_i(\alpha_j)\right)$$

$$= \prod_{i=1}^{N} \zeta_i \tag{40}$$

That is, we can sum first over all the possible states for the single i-th component's energy $\epsilon_i(\alpha_j)$ instead because it allows to express the partition function of the system in terms of the individual partition function of a single particle, given that particles are non-interacting.

$$Z = \frac{1}{h^{3N}} \int dx_1{}^3 \int e^{-\beta \frac{p_1{}^2}{2m_1}} dp_1{}^3 \cdots \int dx_N{}^3 \int e^{-\beta \frac{p_N{}^2}{2m_N}} dp_N{}^3 \tag{41}$$

The solution[4] for ζ_i is obtained by integration of the Gaussian function $\int_{-\infty}^{\infty} e^{-\frac{y^2}{c^2}} dy = |c|\sqrt{\pi}$ in equation (39):

$$\zeta_i = V(2\pi m_i/\beta h^2)^{3/2}$$

$$= V/\lambda_T{}^3 \tag{42}$$

$\lambda_T = \sqrt{2\pi\beta\hbar^2/m_i}$ is the thermal de Broglie wavelength of particle i in equilibrium at temperature T. The argument is that the form of Z due to the independence of particles reduces to sums over independent degrees of freedom; this allows picking which will be done to give a coarser picture of the system. For instance, the system that sums over the last $N/2$ particles' position and momenta (assuming N is even and $m_i = m$ is the same mass for every particle),

[4] Indistinguishable particles require the right side of equation (41) to have a factor of $1/N!$.

$$Z' = \left(\frac{V}{h^3 {\lambda_T}^3}\right)^{N/2} \int dx_1^3 \int e^{-\beta \frac{{p_1}^2}{2m}} dp_1^3 \cdots \int dx_{N/2}^3 \int e^{-\beta \frac{{p_{N/2}}^2}{2m}} dp_{N/2}^3 \tag{43}$$

is not too different to that which sums across the even-indexed particles,

$$Z'' = \left(\frac{V}{h^3 {\lambda_T}^3}\right)^{N/2} \int dx_1^3 \int e^{-\beta \frac{{p_1}^2}{2m}} dp_1^3 \cdots \int dx_{N-1}^3 \int e^{-\beta \frac{{p_{N-1}}^2}{2m}} dp_{N-1}^3 \tag{44}$$

But, at first, it does not look quite the same when only each fourth term is considered

$$Z''' = \left(\frac{V}{h^9 {\lambda_T}^3}\right)^{N/4} \int dx_1^3 \int e^{-\beta \frac{{p_1}^2}{2m_1}} dp_1^3 \cdots \int dx_{N-1}^3 \int e^{-\beta \frac{{p_{N-1}}^2}{2m}} dp_{N-1}^3 \tag{45}$$

(assuming N is divisible by four as well) although they are in fact the same, etc. It is trivial to select in what order we sum, but this shows how the system can have somewhat different representations, which is reflected on a parameter change and the number of degrees of freedom, the aim of renormalization. The suffix "group" is inaccurate however, as the inverse mapping does not exist (an inverse projective coarse graining) for it to be a true group. In this case, the selective map into a system with fewer degrees of freedom only changes the normalization constant.

3.2 The homogeneous 1D Ising model

Now it can be shown how the notion of building block can be adapted to a particular system if we study the renormalization equations for an inhomogeneous system of spins in a closed chain. We will derive the relevant equations first.

Figure 15. A system of atoms represented by their magnetic moment in a lattice which is parallel or anti-parallel to an external magnetic field.

The classical model of Heisenberg for the hamiltonian function of a lattice of unit magnitude spins $\boldsymbol{\sigma_i}$ is:

$$\widetilde{H} = -\sum_{i<j} J_{ij}\vec{\sigma}_i \cdot \vec{\sigma}_j - g\mu_B \sum_i \vec{\sigma}_i \cdot \vec{B} \tag{46}$$

Apart from very low temperatures, quantum mechanical fluctuations are negligible respect to thermal excitations, so classical spins are used to model a ferromagnet. The Ising model further simplifies this idea as it considers spins are not allowed to take any but one specific direction. In this case spins can take two orientations $\boldsymbol{\sigma_i} \rightarrow \sigma_i = \pm 1$.

Further considerations: The one dimensional case with periodic boundary conditions (i.e. a closed chain) will be considered. Interaction parameters J are identical ($J_{ij} = J$ if $j = i \pm 1$; $J_{ij} = 0$ otherwise) thus we say the chain is homogeneous.

The new Hamiltonian function is:

$$\widetilde{H} = -J\sum_i \sigma_i\sigma_{i+1} - h\sum_i \sigma_i - \sum_i C \tag{47}$$

For such a configuration, the partition function[5] is

[5] If $J = 0$, no interactions exist between spins and we find an analogue to the equations (40) of the previous subsection. In this case $\zeta_i = \sum_i \mathrm{e}^{h\sigma_i + c} = \mathrm{e}^c \cosh h$. The partition function for N non-interacting spins in a lattice is then $Z = {\zeta_i}^N = \mathrm{e}^{Nc} \cosh^N H\beta$.

$$Z = \sum_{config} \mathrm{e}^{-\beta \tilde{H}} = \sum_{\sigma_1} \cdots \sum_{\sigma_N} \exp(K \sum_i \sigma_i \sigma_{i+1} + h \sum_i \sigma_i + \sum_i C) \quad (48)$$

Where the first term of the exponential function involves the interaction between adjacent spins with $K = \beta J$, J is a parameter measuring the strength of such interaction; and the second term involves the interaction of each spin with the external magnetic field H, $h = H\beta$. $\beta = \frac{1}{k_B T}$ is the thermodynamic beta associated to the temperature of the system. For now we consider the case of no external field, thus $h = 0$. The third term $C = c\beta$ refers to the point where the zero of energy is located. If there are no interactions between neighbours ($J = 0$) then we write $c = 0$ by convention. We have the partition function:

$$Z = \sum_{\sigma_1} \cdots \sum_{\sigma_N} \exp(K \sum_i \sigma_i \sigma_{i+1}) \quad (49)$$

A transfer matrix as an analogue to the matrices $\mathcal{M}$ and $\mathcal{N}$ which defined the transitions (of the Markov chain) of counters arrays in section 2.3.1 can be defined if we associate it with the conditional probabilities for the states of a spin $\langle \sigma_i |$ to change to spin $|\sigma_{i+1}\rangle$. In this case the transition goes not from a time step to another but from a location in the chain to the next one; hence such matrix $\mathbf{T}$ is defined:

$$\mathbf{T} \equiv \begin{pmatrix} \mathrm{e}^{K} & \mathrm{e}^{-K} \\ \mathrm{e}^{-K} & \mathrm{e}^{K} \end{pmatrix} \quad (50)$$

This transfer matrix can be used on Z by relating a chain element with a neighbour, provided that their elements satisfy:

$$\langle \sigma_i | \mathbf{T} | \sigma_{i+1} \rangle = \mathrm{e}^{K \sigma_i \sigma_{i+1}} \quad (51)$$

So Z is rewritten

$$Z(K) = \sum_{\sigma_1} \cdots \sum_{\sigma_N} \langle \sigma_1 | \mathbf{T} | \sigma_2 \rangle \langle \sigma_2 | \mathbf{T} | \sigma_3 \rangle \cdots \langle \sigma_{N-1} | \mathbf{T} | \sigma_N \rangle \langle \sigma_N | \mathbf{T} | \sigma_1 \rangle \tag{52}$$

This allows to undertake the problem of calculating the partition function by separating it into smaller sub-problems; such is a divide-and-conquer strategy. It is possible to cancel any neighbouring ket-bra pairs as the sum over their states satisfies the discrete completeness resolution of the identity:

$$\sum_j |\sigma_j\rangle\langle\sigma_j| = \mathbf{I} \tag{53}$$

Hence we are able to arbitrarily select which spins we want to show explicitly in the partition function. This is usually done by halving a set of N spins (assuming N is even) as we select even or odd elements to be coarse grained. By applying equivalence (53) to every odd spin, equation (52) yields

$$Z(K) = \sum_{\sigma_2} \sum_{\sigma_4} \cdots \sum_{\sigma_N} \langle \sigma_2 | \mathbf{T}^2 | \sigma_4 \rangle \langle \sigma_4 | \mathbf{T}^2 | \sigma_6 \rangle \cdots \langle \sigma_{N-2} | \mathbf{T}^2 | \sigma_N \rangle \langle \sigma_N | \mathbf{T}^2 | \sigma_2 \rangle \tag{54}$$

The motivation for the use of the renormalization group is that one iterative structure (in this case with the use of a transfer matrix) should be found. Also, that it provides an interpretation centred on the possibility of finding an effective system which looks like the original one. We try to find an expression for $\mathbf{T^2}$ with the same functional form as $\mathbf{T}$. For

$$\mathbf{T'} \equiv \mathbf{T}^2 = \begin{pmatrix} 2\cosh 2K & 2 \\ 2 & 2\cosh 2K \end{pmatrix} \tag{55}$$

to be

$$\mathbf{T'} = f(K') \begin{pmatrix} \mathrm{e}^{K'} & \mathrm{e}^{-K'} \\ \mathrm{e}^{-K'} & \mathrm{e}^{K'} \end{pmatrix}$$

it must be satisfied that

$$K' = \frac{1}{2}\ln\cosh 2K \tag{56.a}$$

$$f(K') = 2\mathrm{e}^{K'} \tag{56.b}$$

So the partition function can now be expressed in terms of both the re-parameterized transformation **T'** and the spin indices $i \to i/2$:

$$Z(K') = \sum_{\sigma_1}\sum_{\sigma_2}\cdots\sum_{\sigma_{N/2}}\langle\sigma_1|\mathbf{T}'|\sigma_2\rangle\langle\sigma_2|\mathbf{T}'|\sigma_3\rangle\cdots\langle\sigma_{(N/2)-1}|\mathbf{T}'|\sigma_{N/2}\rangle\langle\sigma_{N/2}|\mathbf{T}'|\sigma_1\rangle \tag{57}$$

Which has the same form of equation (54), but half of the degrees of freedom originally involved. Equations (56) give the conditions under which an exact coarse-grained representation is attainable for this example of the one-dimensional chain, and allow calculating the effective interactions the now vanished spins represent to their former neighbours. Also, equation (53) permits to repeat this coarse graining and, by selecting the appropriate set of spins, pass from a representation of the partition function $Z(K', N/2)$ to $Z(K'', N/4)$, a system with $N/4$ degrees of freedom whose interaction parameter is given by the iteration of equation (56.a).

The relation between the m-th iteration and its successor is:

$$K^{(m+1)} = \frac{1}{2}\ln\cosh 2K^{(m)} \tag{58}$$

The fixed points for this renormalized system are ${K_1}^* = 0$ and ${K_2}^* \to \infty$. At the first, the Boltzmann factor corresponds to the limit of large temperatures and hence interactions between spins are negligible as a result of disorder whereas ${K_2}^* \to \infty$ is for the absolute zero of temperature, which has dominant interactions between spins hence this point is unstable to perturbations.

3.3 One inhomogeneity in the 1D Ising model

The study of modularity in spin lattices requires the review of those cases that present inhomogeneities. Here we introduce an approach with the methodology of the transfer matrix with the example of one single inhomogeneity and the effect of the renormalization group on such modules.

If we now consider the presence of an interaction between elements i and $i+1$ of the chain which is different to K, say J, we represent it by means of a new transfer matrix, now $\mathbf{M}$. The partition function of the new system is then written:

$$Z(K) = \sum_{\sigma_1} \cdots \sum_{\sigma_i} \cdots \sum_{\sigma_N} \langle \sigma_1 | \mathbf{T} | \sigma_2 \rangle \cdots \langle \sigma_i | \mathbf{M} | \sigma_{i+1} \rangle \cdots \langle \sigma_{N-1} | \mathbf{T} | \sigma_N \rangle \langle \sigma_N | \mathbf{T} | \sigma_1 \rangle \quad (59)$$

An effective interaction between neighbours has to be calculated. By defining the emergent matrix $\mathbf{M'} \equiv \mathbf{TM}$ for i odd, the renormalization group equations are

$$K' = \frac{1}{2} \ln \cosh 2K \quad (60.a)$$

$$J' = \frac{1}{2} \ln \frac{\cosh (K+J)}{\cosh (K-J)} \quad (60.b)$$

The hyperbolic cosine is an even function, then J' remains the same regardless of the matrix multiplication of $\mathbf{T}$ and $\mathbf{M}$, as $\mathbf{M'} = \mathbf{TM} = \mathbf{MT}$ and there is no loss of generality when the even or odd nature of the i-th element concerned is considered.

It follows the analysis of the effects of these equations to the parameters K and J, particularly of the shape of a surface defined $\Delta(K,L) \equiv K'/K - J'/J$. We consider the behaviour of approximations under certain regimes within the (K,J) domain:

i) $K \to 0\ ; J \to 0$

In the limit where the system temperature goes to infinity it is expected a maximal disorder. For K' we have, by expanding the hyperbolic cosines:

$$K' = \frac{1}{2}\ln\cosh 2K$$

$$= \frac{1}{2}\ln\frac{e^{2K}+e^{-2K}}{2}$$

$$= \frac{1}{2}\ln\left(\frac{1}{2}\left(1 + 2K + \frac{(2K)^2}{2} + \cdots + 1 - 2K + \frac{(2K)^2}{2} + \cdots\right)\right)$$

$$\approx \frac{1}{2}\ln(1 + 2K^2)$$

Analogously for J';

$$J' \approx \frac{1}{2}\ln\left(\frac{1+(K+J)^2/2}{1+(K-J)^2/2}\right)$$

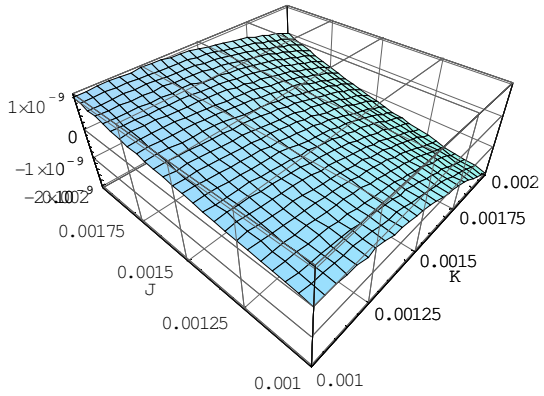
So in both cases, as both $K \to 0$ and $J \to 0$ we further simplify

$$K' \to K^2 \tag{61.a}$$

$$J' \to \frac{1}{4}((K+J)^2 - (K-J)^2) = KJ \tag{61.b}$$

Calculating Δ in these limits yields

$$\Delta_i = 0 \tag{62}$$



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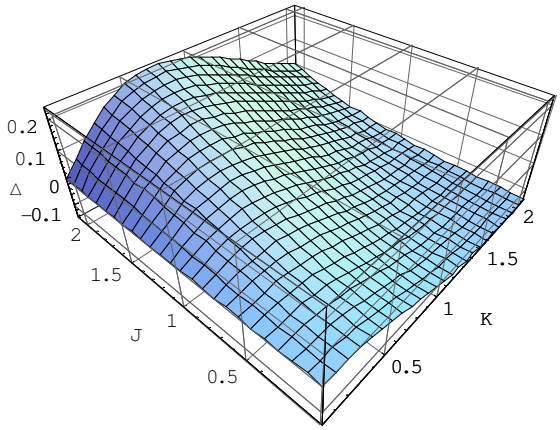

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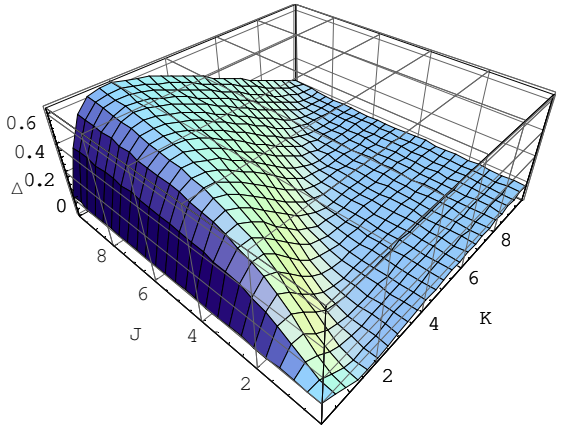

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Figures 16. Effect of the renormalization group equations over a couple of spins with homogeneous interactions (modelled by **T**′) against other with heterogeneous interactions (modelled by **M**′). There is a difference between regimes $K > J$ and $K < J$, which smoothes as both become small.

The sign of Δ has significance for the comparative rescaling of parameters K and J under the renormalization group equations (60); $\Delta_i = 0$ means that in this limit there is no difference in the relative ratio between the new parameters K' and J', hence the inhomogeneity is not any more or less distinct as it was before. The physical interpretation is that the ratio between effective interactions –representative of a spin which is related to its neighbours by interactions of equal magnitude against other where these are different- remains invariant as long as the temperature of the system is high. In this case the renormalization group equations are not able themselves to emphasize or erase the modularity of the system.

It can be thought[16] of J' and K' as illustrating the same original interactions k and j defined before, in a system now at a different temperature T', i.e.

$$K = k\beta \rightarrow K' = k\beta' \tag{63.a}$$

$$J = j\beta \rightarrow J' = j\beta' \tag{63.b}$$

Hence β' refers to a shift in temperature. As $J' < J$ and $K' < K$, the temperature of the new system is higher; for this section we saw that $\Delta_i = 0$, so the interpretation of a positive shift in the temperature via renormalization group is that this limit is stable.

The evolution of parameters through the renormalization group mapping by successive iterations leads to increases of temperature, thus $T \rightarrow \infty$ is the stable fixed point which means a number of renormalization group iterations will lead to make parameters fall under the $\Delta = 0$ scheme.

ii) K large ; $J \ll K$

$$K = \frac{1}{2}\ln\frac{e^{2K}+e^{-2K}}{2}$$

$$= \frac{1}{2}(\ln(e^{2K} + e^{-2K}) - \ln 2)$$

$$J' = \frac{1}{2}\ln\left(\frac{e^{K+J}+e^{-(K+J)}}{e^{K-J}+e^{-(K-J)}}\right)$$

$$= \frac{1}{2}\ln\left(\frac{e^{K+J}\left(1+e^{-2(K+J)}\right)}{e^{K-J}\left(1+e^{-2(K-J)}\right)}\right)$$

$$= \frac{1}{2}\ln\left(e^{2J}\frac{1+e^{-2(K+J)}}{1+e^{-2(K-J)}}\right)$$

$$= \frac{1}{2}\left(2J + \ln\left(\frac{1+e^{-2(K+J)}}{1+e^{-2(K-J)}}\right)\right)$$

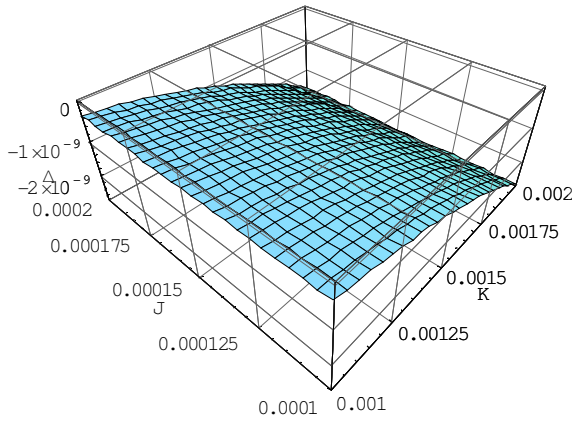
So in both cases, as K is large and $J \ll K$ we further simplify

$$K' \to K - \frac{1}{2}\ln 2 \tag{64.a}$$

$$J' \to J \tag{64.b}$$

Calculating Δ in these limits yields

$$\Delta_{ii} = -\frac{1}{2K}\ln 2 \tag{65}$$



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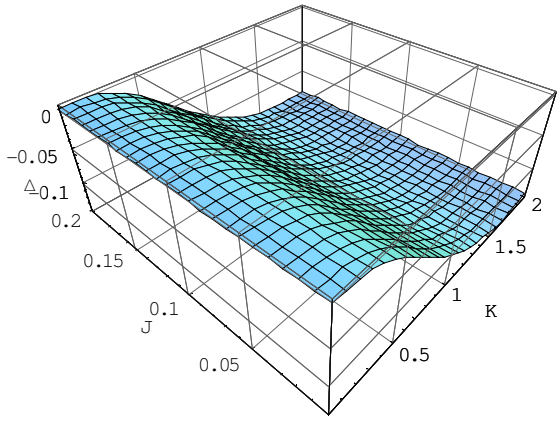

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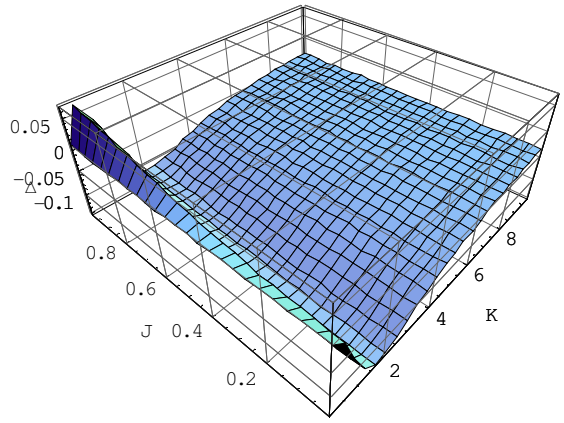

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Figures 17. When under renormalization, an originally stronger interaction K' reduces greatly in comparison with an originally weaker interaction J'. This seems to blur the renormalized system's modularity (scaling: $J = 0.1K$).

So in the limiting case, as $\to \infty$, $\Delta_{ii} \to 0$. The negative sign implies that stronger interactions K under the renormalization group equations are greatly reduced comparatively than the originally much weaker inhomogeneity J. This seems to suggest that the renormalization group homogenises this system in this domain of (K, J). It is interesting to note in figure 17.c a region where Δ is positive, thus emphasising the modularity of the system. This takes place when $J \ll K$ does not hold ($K + J$ and $K - J$ are significantly different).

iii) $J \to \infty$; $K \ll J$

$$J' = \frac{1}{2} \ln \left(\frac{e^{K+J} + e^{-(K+J)}}{e^{K-J} + e^{-(K-J)}} \right)$$

$$= \frac{1}{2} \ln \left(\frac{e^{K+J}\left(1 + e^{-2(K+J)}\right)}{e^{-(K-J)}\left(e^{2(K-J)} + 1\right)} \right)$$

$$= \frac{1}{2} \ln \left(e^{2K} \frac{1 + e^{-2(K+J)}}{1 + e^{-2(-K+J)}} \right)$$

$$= \frac{1}{2}\left(2K + \ln\left(\frac{1+e^{-2(J+K)}}{1+e^{-2(J-K)}}\right)\right)$$

When $J \to \infty$, we simplify

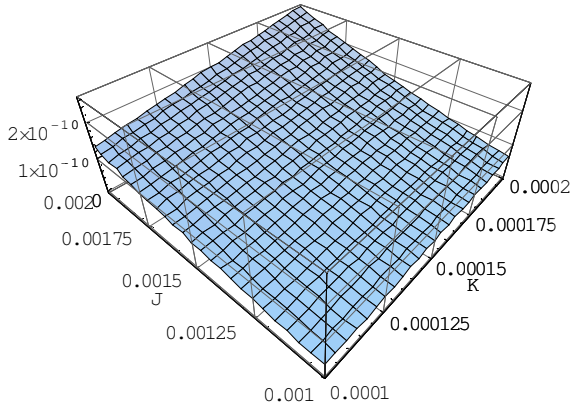
$$J' \to K \tag{66}$$

K remains the same as equation (60.a), so we calculate Δ

$$\Delta_{iii} = K' - \frac{K}{J} \tag{67}$$

Since $K \ll J$, and by inspecting what happens when K is large (see ii)

$$\Delta_{iii} \approx 1 - \frac{1}{2K}\ln 2 \tag{68}$$



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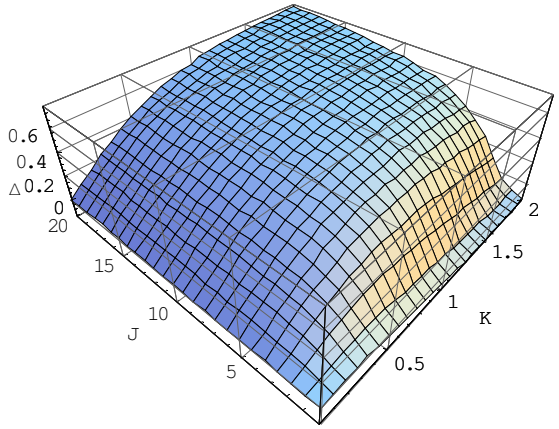

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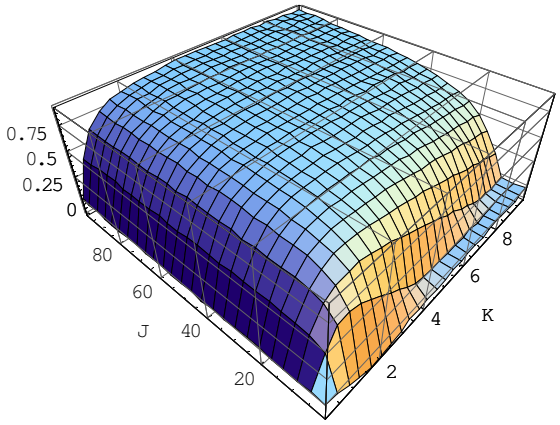

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Figures 18. When under renormalization, an originally weaker interaction K' reduces scarcely in comparison with an originally stronger interaction J'. This seems to blur the renormalized system's modularity (scaling: $K = 0.1J$).

So in the limiting case, as $K \to \infty$, $\Delta_{iii} \to 1$. Unless in the cases where K is very small ($K < ½ \ln 2$) Δ_{iii} is positive, which implies that a stronger inhomogeneity interaction J under the renormalization group equations is greatly reduced comparatively than the originally much weaker interactions K. This seems to suggest that renormalization also homogenises this system in this other domain of (K, J).

We have seen in both low temperature limits that the renormalization group equations do seem to erase the modularity of the system. This might be compatible with the interpretation of a parameterisation that takes a description of the old system to a new one with less degrees of freedom at a higher temperature, hence a modular system is converted

to a more homogeneous one as thermal excitations disorder such arrays. Note that at very low temperatures, quantum mechanical fluctuations should offer different insights to those presented previously, and a correspondence should also exist. As the use of the renormalization group equations seems to homogenise the system studied, it should be useful to quantify how much it does it as well.