Chapter 8
Conclusions and Future Work

In this chapter we will summarize the main contributions of this work according to three main topics:

- Contributions to the Determination of the Configurations associated to the nD-OPP's (**Section 8.1.1**).
- Contributions to Orthogonal Polytopes Topology and Geometry (**Section 8.1.2**).
- Contributions to Orthogonal Polytopes Modeling (**Section 8.1.3**).

Finally, in **Section 8.2**, we will propose some lines of future work. Our proposals are:

- Putting the Odd Edge Characterization in another contexts
- A formal complexity analysis for the nD-EVM
- Relating the nD-EVM with algebraical structures

8.1. Main Contributions

8.1.1. Speeding Up the Determination of the Configurations in the nD-OPP's

According to the results presented in **Chapter 3**, the 'Test-Box' Algorithms perform the identification of the configurations for the nD-OPP's by inspecting a number of combinations which is significantly minor than the total number of possible combinations, while the complexity imposed for determining if two combinations are topologically and geometrically equivalent is reduced by applying our equivalence relations R_E, R_H and R_{adj}. We have showed that our relations are in fact equivalence relations which are "wider" than the equivalence relation based in geometrical transformations, R_f, and therefore they themselves provides an approximate solution to our problem, but when they are combined precisely with relation R_f we speed up the comparison between combinations of hyper-boxes. Moreover, through the configuration's binary representation, both the comparison of combinations and the 'Test-Box' Algorithms have been improved in terms of the time and memory complexity, because an n-dimensional configuration can be managed with only 2^n bits instead of the 2^n vertices for each one of the 2^n possible hyper-boxes. According to our analysis, the best option case of the 'Test-Box' Algorithms corresponds to the recursive definition when a 'Test-Box' is added in the empty hyper-octants embedded in $\mathbb{R}_i^-$. The second best option corresponds to the iterative definition and in third place, according to the number of analyzed combinations, we have the recursive definition when a 'Test-Box" is added in all the empty hyper-octants of a combination.

In **Chapter 3** we also presented an introduction to Pólya's method and we showed how it can be used to compute the total number of configurations for the nD-OPP's. A central concept in this method is the cycle index of the transformations group f^n. The cycle index is a multivariable polynomial that records the cycle structure of each permutation in f^n. The polynomials we have obtained characterize the number of different cases that arise in the configurations of the nD-OPP's. As new contribution, we have reached the counting corresponding to the 5D case.

8.1.2. A New Topological and Geometrical Invariant in the nD-OPP's

In **Chapter 4** we defined some frameworks and equivalence relations in order to demonstrate some properties related with the Odd Edge Characterization and the way it interacts with its incident (n-1)D cells in a combination of nD hyper-boxes. In this sense, Spivak's k-chains have been fruitful in providing the referred frameworks. Spivak's k-chains have allowed us to select, in a unambiguously and formal way, which (n-1)D cells, included in the boundary of a combination of nD hyper-boxes, to consider in order to establish the properties of an Odd or an Even edge from the local point of view of the combinatorial topology in the nD-OPP's.

The concepts of Odd and Even edge were implicitly present in the 1D, 2D, 3D and 4D-OPP's. Because Manifold edges in the 1D, 2D and 3D-OPP's have an odd number of incident segments, rectangles and boxes respectively [Aguilera98]; while Extreme edges in the 4D-OPP's have an odd number of incident 4D hyper-boxes [Pérez-Aguila03b], then all of them can be characterized as Odd Edges. As can be seen, the Odd Edge characterization have provide us an uniform framework based only in the fact of the oddity of the number of incident nD hyper-boxes to a given edge. Because of its properties, the Odd Edge characterization provides us a new topological and geometrical invariant to be considered when we analyze nD-OPP's. In this sense the Odd Edge:

- Provided us an equivalence relation for speeding up the determination of configurations in the nD-OPP's, and
- It had an essential role when we defined the fundaments behind the Extreme Vertices Model in the n-Dimensional space.

8.1.3. The Extension of the Extreme Vertices Model to the n-Dimensional Space for the Modeling of nD-OPP's

In this work we have presented and analyzed the **Extreme Vertices Model in the n-Dimensional Space** (**nD-EVM**). The Extreme Vertices Model allows representing nD-OPP's by means of a single subset of their vertices: the *Extreme Vertices*. As commented in the previous section, the new concept of Odd Edge has a paramount role in the fundamentals of the model. Since the works of Aguilera & Ayala, it is well known that although the EVM of an nD-OPP p has been defined as a subset of the nD-OPP's vertices, there is much more information about p hidden within this subset of vertices. In this work we have extended Aguilera & Ayala's techniques to the nD case in order to obtain this information:

- Computing sections from couplets.
- Computing couplets from sections.
- Computing forward and backward differences of consecutive sections.
- Computing regularized Boolean operations between two nD-OPP's represented through the nD-EVM.
- The conditions for a set of points in nD space to be a valid nD-EVM.

The results obtained leaded us to prove in a formal way, our **Main Hypothesis**:

The Extreme Vertices Model in the n-Dimensional Space (nD-EVM) is a complete scheme for the representation of n-Dimensional Orthogonal Pseudo-Polytopes.

We have proved formally that the following properties are satisfied:

- <u>Domain</u>: The set of objects which are represented in the nD-EVM is clearly the complete set of n-Dimensional Orthogonal Pseudo-Polytopes.
- <u>Validity</u>: We presented conditions which are necessary and sufficient for a finite set of points to be a valid nD-EVM.
- <u>Completeness</u>: We proved that all the geometry, topology and correct boundary orientation of a nD-OPP can be unambiguously obtained from its EVM.
- <u>Uniqueness</u>: For an nD-OPP p its $EVM_n(p)$ is unique.

As expected from any scheme for modeling polytopes, we have experienced the development and performance of some algorithms designed under the context of the Extreme Vertices Model in the n-Dimensional space. We showed the efficiency of our algorithms under the following tasks:

- Regularized Boolean Operations.
- nD-OPP's measures.
- Extraction of boundary elements of nD-OPP's.

The efficiency of such algorithms was evaluated from a statistical point of view. In such statistical analyses we have proposed approximation surfaces that fit as good as possible to the measures we obtained from the execution times of these algorithms. Such surfaces depend on two parameters: the number of input extreme vertices and the number of dimensions. By fixing the number of dimensions in all the equations associated to our approximation surfaces, our functions become dependent only of one variable: the number of input extreme vertices. By this way we have identified that the exponents associated to the number of vertices varies between 1 and 1.5. This complexity for our algorithms provides us elements to determine the temporal efficiency when we perform some operations between nD-OPP's represented through the nD-EVM.

8.2. Future Work

8.2.1. Putting the Odd Edge Characterization in Other Contexts

We have commented the way we approached our study of the Odd Edge Characterizations was based in Spivak's k-chains. The chains are composed by singular general kD hyper-boxes which in **Definition 4.2** are considered as continuous functions from the set $[0, 1]^k$ to a closed set $A \subset \mathbb{R}^n$, that is, a singular general kD hyper-box is denoted by

$$c:[0,1]^k \to A$$

Starting from **Section 4.3** we established that the set A is one of the 2^n sets $[\gamma_1 a_1,\ \gamma_1(a_1+1)]\times\ldots\times[\gamma_n a_n,\ \gamma_n(a_n+1)]$ where $a = (a_1, \ldots, a_n)$ such that $a_i \in \{-1,0\}$, $1 \le i \le n$. In **Definition 4.12** we stated the definition of main edge which at its time leads to the definition of Odd Edge. The main edges where defined in order to be evaluated under sets $[\gamma_1 a_1,\ \gamma_1(a_1+1)]\times\ldots\times[\gamma_n a_n,\ \gamma_n(a_n+1)]$ and for instance our odd edges act on the same sets.

A situation that arises comes from the fact that we could ask if it is possible to adapt our edges characterization to be considered over general closed sets $A \subset \mathbb{R}^n$ that are not necessarily one of $[\gamma_1 a_1,\ \gamma_1(a_1+1)]\times\ldots\times[\gamma_n a_n,\ \gamma_n(a_n+1)]$. That is, it seems interesting to propose the notion of abstract main edge and/or abstract odd edge. Because the odd edge produced an invariant on the nD-OPP's it could be interesting to know if it produces the same impact over other kinds of objects.

Another possible direction considers finding continuous functions between Euclidean space $\mathbb{R}^n$ and other given spaces in such way they preserve the characteristics of odd edges over the nD-OPP's in $\mathbb{R}^n$ and at the same time they provide us the way to analyze and understand other kinds of objects that lie in these given spaces.

8.2.2. Relating the nD-EVM with Algebraical Structures

From **Definition 5.3** we established that an nD-OPP, is an n-chain whose hyper-boxes are in the lattice of $H^n_{L(\gamma_1,\ldots,\gamma_n)}$. Moreover, we stated an nD-OPP is a chain whose set of hyper-boxes is an element of $2^{\left(H^n_{L(\gamma_1,\ldots,\gamma_n)}\right)}$, that is, the power set of $H^n_{L(\gamma_1,\ldots,\gamma_n)}$. Now consider that $EVM_n\left(2^{\left(H^n_{L(\gamma_1,\ldots,\gamma_n)}\right)}\right)$ is the set of Extreme Vertices Models of the nD-OPP's whose associated n-chain is an element of $2^{\left(H^n_{L(\gamma_1,\ldots,\gamma_n)}\right)}$.

Through the above definition we can prove that the set $EVM_n\left(2^{\left(H^n_{L(\gamma_1,\ldots,\gamma_n)}\right)}\right)$ under XOR ($\otimes$) operand forms an Abelian Group. Let $EVM_n(p), EVM_n(q), EVM_n(r) \in EVM_n\left(2^{\left(H^n_{L(\gamma_1,\ldots,\gamma_n)}\right)}\right)$. Then the following properties are satisfied:

- Closure: According to **Theorem 5.19**, $EVM_n(p)\otimes EVM_n(q) = EVM_n(p\otimes{*}q)$ which is also a member of $EVM_n\left(2^{\left(H^n_{L(\gamma_1,\ldots,\gamma_n)}\right)}\right)$ (in fact $p\otimes{*}q$ is a member of $2^{\left(H^n_{L(\gamma_1,\ldots,\gamma_n)}\right)}$).

  $$\therefore \left(\forall EVM_n(p), EVM_n(q) \in EVM_n\left(2^{\left(H^n_{L(\gamma_1,\ldots,\gamma_n)}\right)}\right)\right)\left(EVM_n(p)\otimes EVM_n(q) \in EVM_n\left(2^{\left(H^n_{L(\gamma_1,\ldots,\gamma_n)}\right)}\right)\right)$$

- Associativity:

$$\begin{aligned}
EVM_n(p) \otimes \left(EVM_n(q) \otimes EVM_n(r)\right) &= EVM_n(p) \otimes EVM_n(q \otimes^* r) && \text{(by **Theorem 5.19**)} \\
&= EVM_n\left(p \otimes^* (q \otimes^* r)\right) && \text{(by **Theorem 5.19**)} \\
&= EVM_n\left(\left(p \otimes^* q\right) \otimes^* r\right) && \text{(Because Regularized XOR is associative)} \\
&= EVM_n\left(p \otimes^* q\right) \otimes EVM_n(r) && \text{(by **Theorem 5.19**)} \\
&= \left(EVM_n(p) \otimes EVM_n(q)\right) \otimes EVM_n(r) && \text{(by **Theorem 5.19**)}
\end{aligned}$$

$$\therefore \left(\forall EVM_n(p), EVM_n(q), EVM_n(r) \in EVM_n\left(2^{\left(H^n_{L(\gamma_1,\ldots,\gamma_n)}\right)}\right)\right)$$
$$\left(\left(EVM_n(p) \otimes EVM_n(q)\right) \otimes EVM_n(r) = EVM_n(p) \otimes \left(EVM_n(q) \otimes EVM_n(r)\right)\right)$$

- Existence of the identical element:

  Consider the null polytope $\varnothing \in 2^{\left(H^n_{L(\gamma_1,\ldots,\gamma_n)}\right)}$, hence $EVM_n(\varnothing) = \varnothing \in EVM_n\left(2^{\left(H^n_{L(\gamma_1,\ldots,\gamma_n)}\right)}\right)$. Therefore we have

$$EVM_n(p) \otimes EVM_n(\varnothing) = EVM_n(p) \otimes \varnothing = \varnothing \otimes EVM_n(p) = EVM_n(p)$$

$$\therefore \left(\exists EVM_n(\varnothing) = \varnothing \in EVM_n\left(2^{\left(H^n_{L(\gamma_1,\ldots,\gamma_n)}\right)}\right)\right)$$
$$\left(EVM_n(p) \otimes EVM_n(\varnothing) = EVM_n(\varnothing) \otimes EVM_n(p) = EVM_n(p),\ \forall EVM_n(p) \in EVM_n\left(2^{\left(H^n_{L(\gamma_1,\ldots,\gamma_n)}\right)}\right)\right)$$

- Existence of the inverse element for each element in $EVM_n\left(2^{\left(H^n_{L(\gamma_1,\ldots,\gamma_n)}\right)}\right)$:

$$\begin{aligned}
EVM_n(p) \otimes EVM_n(p) &= EVM_n\left(p \otimes^* p\right) && \text{(by **Theorem 5.19**)} \\
&= EVM_n\left(\left(p -^* p\right) \cup^* \left(p -^* p\right)\right) && \text{(by definition of regularized XOR operator)} \\
&= EVM_n\left(\varnothing \cup \varnothing\right) && \text{(Because the regularized difference between p and p is the empty set)} \\
&= EVM_n(\varnothing)
\end{aligned}$$

  That is, each element in $EVM_n\left(2^{\left(H^n_{L(\gamma_1,\ldots,\gamma_n)}\right)}\right)$ is its own inverse element.

$$\therefore \left(\forall EVM_n(p) \in EVM_n\left(2^{\left(H^n_{L(\gamma_1,\ldots,\gamma_n)}\right)}\right)\right)\left(EVM_n(p) \otimes EVM_n(p) = EVM_n(\varnothing)\right)$$

- Commutativity:

$$\begin{aligned}
EVM_n(p) \otimes EVM_n(q) &= EVM_n(p \otimes^* q) && \text{(by **Theorem 5.19**)} \\
&= EVM_n(q \otimes^* p) && \text{(Because regularized XOR is commutative)} \\
&= EVM_n(q) \otimes EVM_n(p) && \text{(by **Theorem 5.19**)}
\end{aligned}$$

$$\therefore \left(\forall EVM_n(p), EVM_n(q) \in EVM_n\left(2^{\left(H^n_{L(\gamma_1,\ldots,\gamma_n)}\right)}\right)\right)\left(EVM_n(p) \otimes EVM_n(q) = EVM_n(q) \otimes EVM_n(p)\right)$$

Because the above properties are satisfied then we have that the nD-EVM composes an Abelian Group under the Xor operator. It is natural to ask if the nD-EVM can compose other Algebraical Structures such as Rings, Fields, Vector Spaces, Distributive Lattices, etc. It is clear that in some cases some operations should be defined, for example, vector additions and scalar product if we are considering a Vector Space. By characterizing the EVM as one of these structures we will be opening new frontiers in the sense that we will know that a great amount of well known properties associated to such structures are, for instance, valid for the EVM. By this way new problems could be modeled by taking in account all the advantages provided by previously knowing these properties.

8.2.3. A Formal Complexity Analysis for the nD-EVM

According to the results obtained in **Chapter 6**, which are summarized in **Table 8.1**, our experimental execution times have provided us clues about the bounds that should be obtained when a formal complexity analysis is applied to the nD-EVM. As stated in **Chapter 6**, in this work we have considered the time complexity topic from a statistical point of view. The bounds we provided have been obtained from experimental data which were obtained according to procedures we mentioned with detail in the corresponding sections.

Algorithm	Operation	Approximation Surface	x's exponent	R^2
6.4	Regularized Intersection	$t = 4{,}271.11\ x^{1.1737}n^{1.0862}$	1.1737	0.9234
6.4	Regularized Union	$t = 16{,}698.63\ x^{1.0821}n^{1.0607}$	1.0821	0.9221
6.4	Regularized Xor	$t = 483.17\ x^{1.4161}n^{1.4161} + 108{,}263{,}080$	1.4161	0.9260
6.5	Computing Content	$t = 4{,}763.939\ x^{1.1894}n^{0.8390}$	1.1894	0.9803
6.6	Computing Boundary Content	$t = \dfrac{12921.02\ x + 1454430\ n - 7196150}{0.0212336\ n^3 - 0.247936\ n^2 + 0.92776\ n - 1}$	1.0000	0.9963
6.7	Extracting Forward and Backward Differences	$t = 1{,}849.27\ x^{1.3}n^{1.64855}$	1.3000	0.9797
6.8	Building Differences Tree	$t = 1{,}160.7\ x^{1.3189}n^{2.3720}$	1.3189	0.9871

Table 8.1. Experimental execution times of algorithms under the nD-EVM (x: Number of input extreme vertices, n: number of dimensions).

Because our algorithms are recursive on the number of dimensions of the input polytopes and in each recursivity level a wide range of situations can be present is that a formal analysis for time complexity should consider these issues. In [Aguilera98], under the context of the 2D and 3D-EVM, worst case complexities for some algorithms were presented and discussed. In [Rodriguez04] some modifications to Aguilera's algorithms were applied in order to perform some operations under the 3D-EVM in a faster way. A formal treatment of time complexity should start by considering the formal results presented in [Aguilera98], the modifications suggested in [Rodriguez04]; and our experimental results. We expect the estimations presented in this work can be useful in suggesting time bounds to obtain in worst, best and average cases.