Chapter 5
Orthogonal Polytopes Modeling Through the Extreme Vertices Model in the n-Dimensional Space (nD-EVM)

The Extreme Vertices Model (3D-EVM) was originally presented by Aguilera & Ayala in [Aguilera97] (for representing only 2-manifold Orthogonal Polyhedra) and widely described in [Aguilera98] (considering both Orthogonal Polyhedra and Pseudo-Polyhedra). This model has enabled the development of simple and robust algorithms for performing the most usual and demanding tasks on solid modeling, such as closed and regularized Boolean operations, solid splitting, set membership classification operations and measure operations on 3D-OPP's Later on, in [Rodriguez04] was presented an Enriched-EVM. As proposed in [Aguilera98], it is natural to ask if the EVM can be extended for modeling nD-OPP's. In this sense, some experiments have been made, in [Pérez-Aguila03b] and [Pérez-Aguila03d], where the validity of the model was assumed true in order to represent 4D and 5D-OPP's. The results obtained leaded us to this chapter, where we will prove in a formal way, the Main Hypothesis of this work:

The Extreme Vertices Model in the n-Dimensional Space (nD-EVM) is a complete scheme for the representation of n-Dimensional Orthogonal Pseudo-Polytopes.

In **Section 5.1** we will introduce some conventions and preliminary background related directly with nD-OPP's. In **Section 5.2** we will establish the foundations of the nD-EVM. It will be seen how the Odd Edge Topological Characterization in the nD-OPP's has a paramount role in this last aspect. As seen in the previous chapter, we will deal with Local and Global Analysis over the nD-OPP's but now under the context of the nD-EVM (**Sections 5.3** and **5.4**). Finally, in **Sections 5.5** to **5.7** the concepts and results originally presented by Aguilera & Ayala, in [Aguilera97] and [Aguilera98], will be presented and discussed under the new context of the nD-EVM.

5.1. Preliminary Background

Definition 5.1: *Consider a lattice* $L^n_{(\gamma_1,\ldots,\gamma_n)}$. *Let* $\mathbf{p} = (p_1, \ldots, p_n)$ *a point in* $L^n_{(\gamma_1,\ldots,\gamma_n)}$. *We define to the* *<u>general singular nD hyper-box associated to **p**</u>* *as the function:*

$$\begin{array}{rccl} c: & [0,1]^n & \to & [p_1, p_1+\gamma_1]\times\ldots\times[p_n, p_n+\gamma_n] \\ & x & \sim & c(x) = (\gamma_1 x_1 + p_1, \ldots, \gamma_n x_n + p_n) \end{array}$$

For example, consider lattice $L^2_{(4,6)}$. Let c be the general singular rectangle associated to the point $\mathbf{p} = (4, 12)$ (See **Figure 5.1**):

$$\begin{array}{rccl} c: & [0,1]^2 & \to & [4,6]\times[12,18] \\ & x & \sim & c(x) = (4x_1+4, 6x_2+12) \end{array}$$

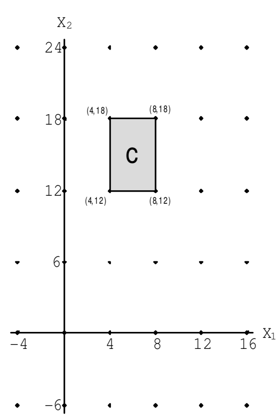


Figure 5.1. The rectangle associated to the point $\mathbf{p} = (4, 12) \in L^2_{(4,6)}$.

Definition 5.2 [Jonas95]: *Let* $H^n_{L_{(\gamma_1,\ldots,\gamma_n)}}$ *be defined as the set of all the general singular nD hyper-boxes associated to the points of the lattice* $L^n_{(\gamma_1,\ldots,\gamma_n)}$.

Definition 5.3 [Jonas95]: *We will say that an* <u>*n-Dimensional Orthogonal Pseudo-Polytope*</u>*, or just an* <u>*nD-OPP*</u>*, will be an n-chain whose hyper-boxes are in* $H^n_{L_{(\gamma_1,\ldots,\gamma_n)}}$*. In fact, an nD-OPP is a chain whose set of hyper-boxes is an element of* $2^{\left(H^n_{L_{(\gamma_1,\ldots,\gamma_n)}}\right)}$*, that is, the power set of* $H^n_{L_{(\gamma_1,\ldots,\gamma_n)}}$.

Some of the results to be presented will require dealing with the hyper-boxes that compose an nD-OPP or they will require to consider subsets of points in $\mathbb{R}^n$ which belong to the nD-OPP. In this last sense, the points that compose to an nD-OPP are obtained through the union of the images of the hyper-boxes in its corresponding n-chain. In the other hand, according to **Definition 4.10**, the boundary of an nD-OPP p is the (n-1)-chain ∂(p) which is composed by the (n-1)D cells that are not shared by two hyper-boxes of p. The vanishing of the shared (n-1)D cells is given when the sum of their orientations is zero.

For example, consider the 2D-OPP q shown in **Figure 5.2** which is described by a set of 2D boxes under lattice $L^2_{(5,2)}$. Its corresponding rectangles and their associated points in lattice $L^2_{(5,2)}$ are described in **Table 5.1**.

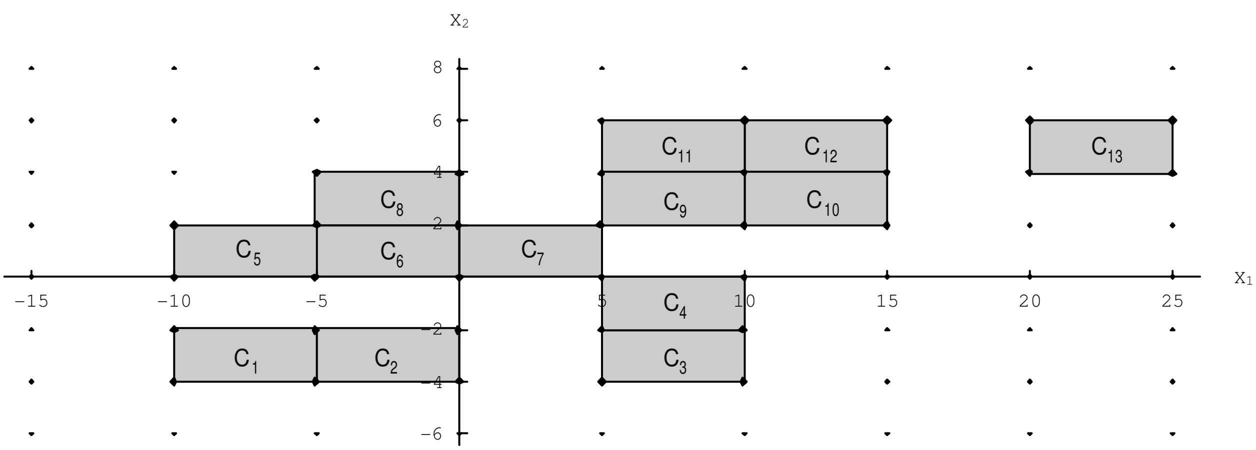


Figure 5.2. A 2D-OPP q described by a set of rectangles under lattice $L^2_{(5,2)}$.

Associated Point	Rectangle	Associated Point	Rectangle
p₁ = (-10, -4)	$c_1: [0,1]^2 \to [-10,-5]\times[-4,-2]$ $x \sim c_1(x)=(5x_1-10, 2x_2-4)$	**p₈** = (-5, 2)	$c_8: [0,1]^2 \to [-5,0]\times[2,4]$ $x \sim c_8(x)=(5x_1-5, 2x_2+2)$
p₂ = (-5, -4)	$c_2: [0,1]^2 \to [-5,0]\times[-4,-2]$ $x \sim c_2(x)=(5x_1-5, 2x_2-4)$	**p₉** = (5, 2)	$c_9: [0,1]^2 \to [5,10]\times[2,4]$ $x \sim c_9(x)=(5x_1+5, 2x_2+2)$
p₃ = (5, -4)	$c_3: [0,1]^2 \to [5,10]\times[-4,-2]$ $x \sim c_3(x)=(5x_1+5, 2x_2-4)$	**p₁₀** = (10, 2)	$c_{10}: [0,1]^2 \to [10,15]\times[2,4]$ $x \sim c_{10}(x)=(5x_1+10, 2x_2+2)$
p₄ = (5, -2)	$c_4: [0,1]^2 \to [5,10]\times[-2,0]$ $x \sim c_4(x)=(5x_1+5, 2x_2-2)$	**p₁₁** = (5, 4)	$c_{11}: [0,1]^2 \to [5,10]\times[4,6]$ $x \sim c_{11}(x)=(5x_1+5, 2x_2+4)$
p₅ = (-10, 0)	$c_5: [0,1]^2 \to [-10,-5]\times[0,2]$ $x \sim c_5(x)=(5x_1-10, 2x_2)$	**p₁₂** = (10, 4)	$c_{12}: [0,1]^2 \to [10,15]\times[4,6]$ $x \sim c_{12}(x)=(5x_1+10, 2x_2+4)$
p₆ = (-5, 0)	$c_6: [0,1]^2 \to [-5,0]\times[0,2]$ $x \sim c_6(x)=(5x_1-5, 2x_2)$	**p₁₃** = (20, 4)	$c_{13}: [0,1]^2 \to [20,25]\times[4,6]$ $x \sim c_{13}(x)=(5x_1+20, 2x_2+4)$
p₇ = (0, 0)	$c_7: [0,1]^2 \to [0,5]\times[0,2]$ $x \sim c_7(x)=(5x_1, 2x_2)$		

Table 5.1. The rectangles that compose the 2D-OPP q shown in **Figure 5.2**.

Hence, the 2-chain associated to q is given by:

$$
\begin{array}{lllllllll}
q = & (5x_1 - 10, 2x_2 - 4) & + & (5x_1 - 5, 2x_2 - 4) & + & (5x_1 + 5, 2x_2 - 4) & + & (5x_1 + 5, 2x_2 - 2) & + \\
 & (5x_1 - 10, 2x_2) & + & (5x_1 - 5, 2x_2) & + & (5x_1, 2x_2) & + & (5x_1 - 5, 2x_2 + 2) & + \\
 & (5x_1 + 5, 2x_2 + 2) & + & (5x_1 + 10, 2x_2 + 2) & + & (5x_1 + 5, 2x_2 + 4) & + & (5x_1 + 10, 2x_2 + 4) & + \\
 & (5x_1 + 20, 2x_2 + 4) & & & & & & &
\end{array}
$$

The boundary edges associated to the rectangles in q are shown in the **Table 5.2**.

Rectangle	**Boundary Edges**	**Orientation**	**Rectangle**	**Boundary Edges**	**Orientation**
$c_1(x) = (5x_1 - 10, 2x_2 - 4)$	$(-10, 2x_1 - 4)$	-1	$c_8(x) = (5x_1 - 5, 2x_2 + 2)$	$(-5, 2x_1 + 2)$	-1
	$(-5, 2x_1 - 4)$	1		$(0, 2x_1 + 2)$	1
	$(5x_1 - 10, -4)$	1		$(5x_1 - 5, 2)$	1
	$(5x_1 - 10, -2)$	-1		$(5x_1 - 5, 4)$	-1
$c_2(x) = (5x_1 - 5, 2x_2 - 4)$	$(-5, 2x_1 - 4)$	-1	$c_9(x) = (5x_1 + 5, 2x_2 + 2)$	$(5, 2x_1 + 2)$	-1
	$(0, 2x_1 - 4)$	1		$(10, 2x_1 + 2)$	1
	$(5x_1 - 5, -4)$	1		$(5x_1 + 5, 2)$	1
	$(5x_1 - 5, -2)$	-1		$(5x_1 + 5, 4)$	-1
$c_3(x) = (5x_1 + 5, 2x_2 - 4)$	$(5, 2x_1 - 4)$	-1	$c_{10}(x) = (5x_1 + 10, 2x_2 + 2)$	$(10, 2x_1 + 2)$	-1
	$(10, 2x_1 - 4)$	1		$(15, 2x_1 + 2)$	1
	$(5x_1 + 5, -4)$	1		$(5x_1 + 10, 2)$	1
	$(5x_1 + 5, -2)$	-1		$(5x_1 + 10, 4)$	-1
$c_4(x) = (5x_1 + 5, 2x_2 - 2)$	$(5, 2x_1 - 2)$	-1	$c_{11}(x) = (5x_1 + 5, 2x_2 + 4)$	$(5, 2x_1 + 4)$	-1
	$(10, 2x_1 - 2)$	1		$(10, 2x_1 + 4)$	1
	$(5x_1 + 5, -2)$	1		$(5x_1 + 5, 4)$	1
	$(5x_1 + 5, 0)$	-1		$(5x_1 + 5, 6)$	-1
$c_5(x) = (5x_1 - 10, 2x_2)$	$(-10, 2x_1)$	-1	$c_{12}(x) = (5x_1 + 10, 2x_2 + 4)$	$(10, 2x_1 + 4)$	-1
	$(-5, 2x_1)$	1		$(15, 2x_1 + 4)$	1
	$(5x_1 - 10, 0)$	1		$(5x_1 + 10, 4)$	1
	$(5x_1 - 10, 2)$	-1		$(5x_1 + 10, 6)$	-1
$c_6(x) = (5x_1 - 5, 2x_2)$	$(-5, 2x_1)$	-1	$c_{13}(x) = (5x_1 + 20, 2x_2 + 4)$	$(20, 2x_1 + 4)$	-1
	$(0, 2x_1)$	1		$(25, 2x_1 + 4)$	1
	$(5x_1 - 5, 0)$	1		$(5x_1 + 20, 4)$	1
	$(5x_1 - 5, 2)$	-1		$(5x_1 + 20, 6)$	-1
$c_7(x) = (5x_1, 2x_2)$	$(0, 2x_1)$	-1			
	$(5, 2x_1)$	1			
	$(5x_1, 0)$	1			
	$(5x_1, 2)$	-1			

Table 5.2. The boundary edges of the rectangles that compose the 2D-OPP q shown in **Figure 5.2**.

Therefore, q's boundary is given by the 1-chain:

$$
\begin{array}{lllllllll}
\partial(q) = & -(-10, 2x_1 - 4) & + & (5x_1 - 10, -4) & - & (5x_1 - 10, -2) & + & (0, 2x_1 - 4) & + \; (5x_1 - 5, -4) \\
 & -(5x_1 - 5, -2) & - & (5, 2x_1 - 4) & + & (10, 2x_1 - 4) & + & (5x_1 + 5, -4) & - \; (5, 2x_1 - 2) \\
 & +(10, 2x_1 - 2) & - & (5x_1 + 5, 0) & - & (-10, 2x_1) & + & (5x_1 - 10, 0) & - \; (5x_1 - 10, 2) \\
 & +(5x_1 - 5, 0) & + & (5, 2x_1) & + & (5x_1, 0) & - & (5x_1, 2) & - \; (-5, 2x_1 + 2) \\
 & +(0, 2x_1 + 2) & - & (5x_1 - 5, 4) & - & (5, 2x_1 + 2) & + & (5x_1 + 5, 2) & + \; (15, 2x_1 + 2) \\
 & +(5x_1 + 10, 2) & - & (5, 2x_1 + 4) & - & (5x_1 + 5, 6) & + & (15, 2x_1 + 4) & - \; (5x_1 + 10, 6) \\
 & -(20, 2x_1 + 4) & + & (25, 2x_1 + 4) & + & (5x_1 + 20, 4) & - & (5x_1 + 20, 6) &
\end{array}
$$

By **Theorem 4.1**, for a given nD-OPP p we have that $\partial(\partial(p)) = 0$. However, there is a procedure to have access to the kD elements on the boundary of an nD hyper-box, $0 \leq k < (n-1)$. Let's extract, as an example, the vertices ($k = 0$) of a 3D singular box (a unitary cube).

Consider the following singular 1D hyper-box:

$$
\begin{array}{rcl}
I^1: \; [0,1] & \rightarrow & [0,1] \\
x & \sim & I^1(x) = x
\end{array}
$$

Its corresponding 0D-cells, its vertices, according to **Definition 4.3**, are given by:

$$I^1_{(1,0)}(x) = I^1(0) = (0)$$

$$I^1_{(1,1)}(x) = I^1(1) = (1)$$

Now consider a singular 2D hyper-box:

$$\begin{array}{llll} I^2: & [0,1]^2 & \rightarrow & [0,1]^2 \\ & x & \sim & I^2(x) = x \end{array}$$

Whose corresponding edges, according to **Definition 4.3**, are given by:

$$I^2_{(1,0)}(x_1) = I^2(0,x_1) = (0,x_1)$$

$$I^1_{(1,1)}(x_1) = I^2(1,x_1) = (1,x_1)$$

$$I^2_{(2,0)}(x_1) = I^2(x_1,0) = (x_1,0)$$

$$I^1_{(2,1)}(x_1) = I^2(x_1,1) = (x_1,1)$$

By evaluating the vertices of the 1D segment in the edges of the 2D cube we obtain the pair of vertices that define each edge of the 2D cube:

Edge $I^2_{(1,0)}(x_1) = (0,x_1)$

- $I^2_{(1,0)}(I^1_{(1,0)}(x)) = I^2_{(1,0)}(0) = (0,0)$
- $I^2_{(1,0)}(I^1_{(1,1)}(x)) = I^2_{(1,0)}(1) = (0,1)$

Edge $I^1_{(1,1)}(x_1) = (1,x_2)$

- $I^2_{(1,1)}(I^1_{(1,0)}(x)) = I^2_{(1,1)}(0) = (1,0)$
- $I^2_{(1,1)}(I^1_{(1,1)}(x)) = I^2_{(1,1)}(1) = (1,1)$

Edge $I^2_{(2,0)}(x_1) = (x_1,0)$

- $I^2_{(2,0)}(I^1_{(1,0)}(x)) = I^2_{(2,0)}(0) = (0,0)$
- $I^2_{(2,0)}(I^1_{(1,1)}(x)) = I^2_{(2,0)}(1) = (1,0)$

Edge $I^1_{(2,1)}(x_1) = (x_1,1)$

- $I^2_{(2,1)}(I^1_{(1,0)}(x)) = I^2_{(2,1)}(0) = (0,1)$
- $I^2_{(2,1)}(I^1_{(1,1)}(x)) = I^2_{(2,1)}(1) = (1,1)$

Finally, consider the singular 3D hyper-box:

$$\begin{array}{llll} I^3: & [0,1]^3 & \rightarrow & [0,1]^3 \\ & x & \sim & I^3(x) = x \end{array}$$

Whose faces are defined by:

$$I^3_{(1,0)}(x) = I^3(0,x_1,x_2) = (0,x_1,x_2)$$

$$I^3_{(1,1)}(x) = I^3(1,x_1,x_2) = (1,x_1,x_2)$$

$$I^3_{(2,0)}(x) = I^3(x_1,0,x_2) = (x_1,0,x_2)$$

$$I^1_{(2,1)}(x) = I^3(x_1,1,x_2) = (x_1,1,x_2)$$

$$I^3_{(3,0)}(x) = I^3(x_1,x_2,0) = (x_1,x_2,0)$$

$$I^1_{(3,1)}(x) = I^3(x_1,x_2,1) = (x_1,x_2,1)$$

By evaluating the vertices of the edges in the 2D cube we obtain the quartets of vertices that define each face of the 3D cube:

Face $I^3_{(1,0)}(x) = (0,x_1,x_2)$

- $I^3_{(1,0)}(0,0) = (0,0,0)$
- $I^3_{(1,0)}(0,1) = (0,0,1)$
- $I^3_{(1,0)}(1,0) = (0,1,0)$
- $I^3_{(1,0)}(1,1) = (0,1,1)$

Face $I^3_{(1,1)}(x) = (1,x_1,x_2)$

- $I^3_{(1,1)}(0,0) = (1,0,0)$
- $I^3_{(1,1)}(0,1) = (1,0,1)$
- $I^3_{(1,1)}(1,0) = (1,1,0)$
- $I^3_{(1,1)}(1,1) = (1,1,1)$

Face $I^3_{(2,0)}(x) = I^3(x_1,0,x_2) = (x_1,0,x_2)$

- $I^3_{(2,0)}(0,0) = (0,0,0)$
- $I^3_{(2,0)}(0,1) = (0,0,1)$
- $I^3_{(2,0)}(1,0) = (1,0,0)$
- $I^3_{(2,0)}(1,1) = (1,0,1)$

Face $I^1_{(2,1)}(x) = (x_1,1,x_2)$

- $I^3_{(2,1)}(0,0) = (0,1,0)$
- $I^3_{(2,1)}(0,1) = (0,1,1)$
- $I^3_{(2,1)}(1,0) = (1,1,0)$
- $I^3_{(2,1)}(1,1) = (1,1,1)$

Face $I^3_{(3,0)}(x) = (x_1,x_2,0)$

- $I^3_{(3,0)}(0,0) = (0,0,0)$
- $I^3_{(3,0)}(0,1) = (0,1,0)$
- $I^3_{(3,0)}(1,0) = (1,0,0)$
- $I^3_{(3,0)}(1,1) = (1,1,0)$

Face $I^1_{(3,1)}(x) = (x_1,x_2,1)$

- $I^3_{(3,1)}(0,0) = (0,0,1)$
- $I^3_{(3,1)}(0,1) = (0,1,1)$
- $I^3_{(3,1)}(1,0) = (1,0,1)$
- $I^3_{(3,1)}(1,1) = (1,1,1)$

According to this procedure, in order to obtain the kD elements on the boundary of an nD hyper-box, we must extract the boundary elements of a (k+1)D hyper-box. Those boundary elements are evaluated in the boundary elements of a (k+2)D hyper-box and so on until we evaluate the resulting compositions of the previous evaluations over the boundary elements of the nD hyper-box.

In previous sections we have defined lattices in $\mathbb{R}^n$, in such way that an nD-OPP is described through nD hyper-boxes associated to points in a lattice. Some applications can find this scheme restrictive in the sense that coordinates in lattices are equally spaced along their axes. Before going any further we will describe a procedure to define point grids in $\mathbb{R}^n$ given the vertices of an nD-OPP. Such grids will have their coordinates not necessarily equally spaced along their axes. Moreover, we will see how our definitions for lattices are applicable in this context.

Definition 5.4. *Consider the set of vertices V(p) of an nD-OPP. Let* $\overline{V_i(p)=\{w_1,w_2,\ldots,w_{nx_i}\}}$ *be the set of different values such that* $w_j < w_{j+1}$ *and* w_j *is the coordinate associated to* X_i*-axis of some vertex* $V_k \in V(p)$ *where* $1 \le j \le nx_i$ *and* $1 \le k \le Card(V(p))$*.*

Consider for example the 2D-OPP's p and q shown in **Figures 5.3.a** and **5.3.b** respectively. We have that $V_1(p) = \{1, 3, 4, 6\}$ and $V_2(p) = \{1, 3, 5\}$, while $V_1(q) = \{1, 2, 5\}$ and $V_2(q) = \{1, 2, 4\}$.

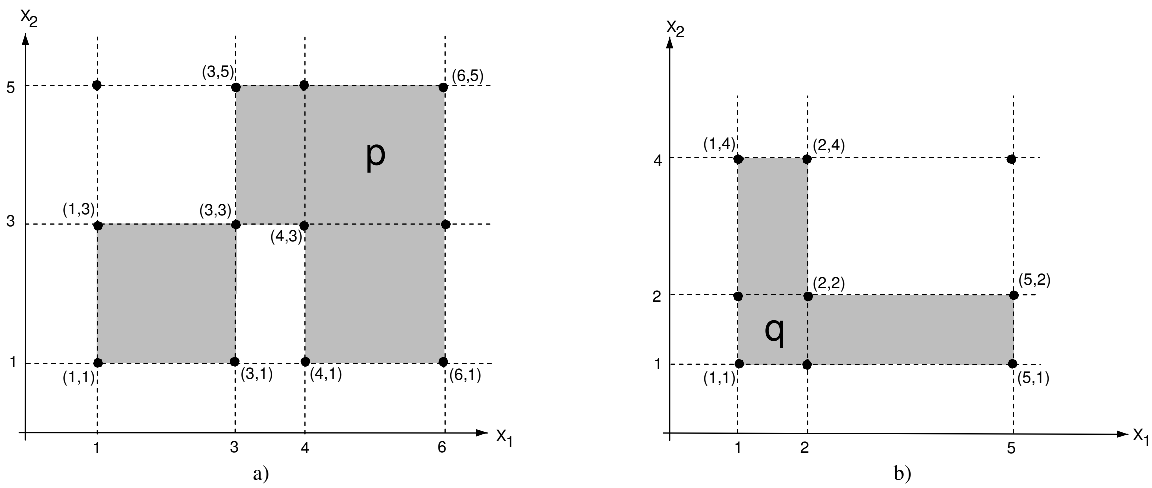


Figure 5.3. Two 2D-OPP's p and q. a) 2D-OPP p and its associated point grid.
b) 2D-OPP q and its associated point grid (See text for details).

By computing the Cartesian product $V_1(p) \times V_2(p)$ we obtain the finite point grid

$$\{(1,1), (1,3), (1,5), (3,1), (3,3), (3,5), (4,1), (4,3), (4,5), (6,1), (6,3), (6,5)\}$$

Through the product $V_1(q) \times V_2(q)$ we have

$$\{(1,1), (1,2), (1,4), (2,1), (2,2), (2,4), (5,1), (5,2), (5,4)\}$$

Figures 5.3.a and **5.3.b** show the generated grids and the disposition of the exemplified 2D-OPP's in such grids. In more general terms, we have that an nD-OPP p with their respective sets $V_1(p)$, $V_2(p)$, …, $V_n(p)$ will have associated the finite n-dimensional point grid given by $V_1(p) \times V_2(p) \times \ldots \times V_n(p)$.

Now, let's proceed to describe our exemplified 2D-OPP's as a union of general singular 2D hyper-boxes, i.e., rectangles whose vertices coincide with points in the associated grid to the OPP's. According to **Definition 5.1** all points in a lattice will have associated one nD hyper-box. In our new context we can establish a similar definition: a point $v = (v_1, \ldots, v_n)$ in a grid associated to an nD-OPP p will have its corresponding general singular nD hyper-box c as follows:

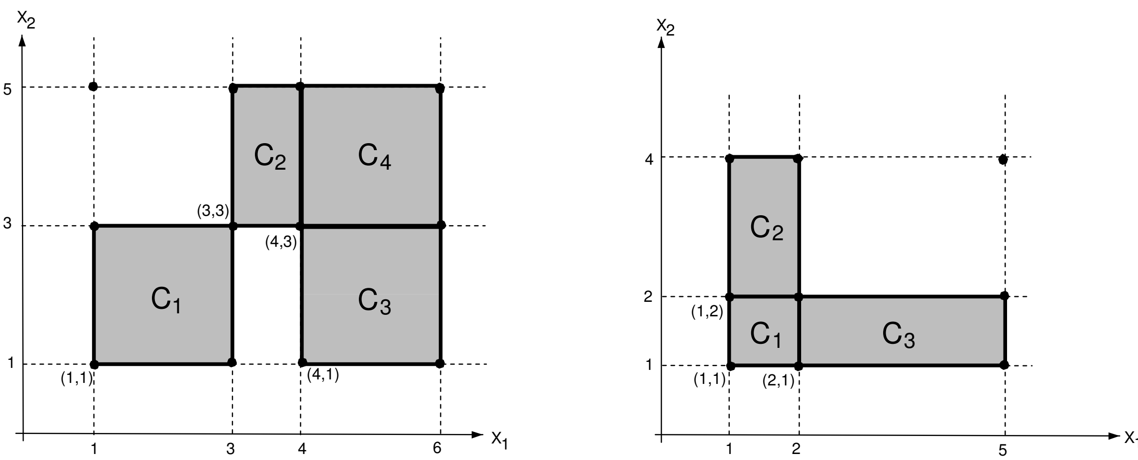
$$\begin{array}{rccc} c: & [0,1]^n & \to & [v_1,v_1']\times\ldots\times[v_n,v_n'] \\ & x & \sim & c(x)=((v_1'-v_1)x_1+v_1,\ldots,(v_n'-v_n)x_n+v_n) \end{array}$$

Where v_i' is the next coordinate following to v_i in $V_i(p)$, $1 \le i \le n$. We will say that the nD hyper-box c corresponding to point v in the grid associated to an nD-OPP p is in such OPP if and only if

$$c([0,1]^n) \subseteq p$$

According to the previous definitions a subset of V(p) will define the nD hyper-boxes that describe to p under its associated grid.

For example, the 2D-OPP's p and q shown in **Figures 5.3.a** and **5.3.b** are now defined by the rectangles described in **Table 5.3**. The **Figures 5.4.a** and **5.4.b** show the disposition of those rectangles in the grids associated to p and q respectively.



a) b)

Figure 5.4. The 2D-OPP's from **Figure 5.3** described through rectangles associated to points corresponding to their grids. Only the points that describe the rectangles are shown.

2D-OPP p		**2D-OPP q**	
Point	**Rectangle**	**Point**	**Rectangle**
(1,1)	$c_1: [0,1]^2 \to [1,3]\times[1,3]$ $x \sim c_1(x)=(2x_1+1, 2x_2+1)$	(1,1)	$c_1: [0,1]^2 \to [1,2]\times[1,2]$ $x \sim c_1(x)=(x_1+1, x_2+1)$
(3,3)	$c_2: [0,1]^2 \to [3,4]\times[3,5]$ $x \sim c_2(x)=(x_1+3, 2x_2+3)$	(1,2)	$c_2: [0,1]^2 \to [1,2]\times[2,4]$ $x \sim c_2(x)=(x_1+1, 2x_2+2)$
(4,1)	$c_3: [0,1]^2 \to [4,6]\times[1,3]$ $x \sim c_3(x)=(2x_1+4, 2x_2+1)$	(2,1)	$c_3: [0,1]^2 \to [2,5]\times[1,2]$ $x \sim c_3(x)=(3x_1+2, x_2+1)$
(4,3)	$c_4: [0,1]^2 \to [4,6]\times[3,5]$ $x \sim c_4(x)=(2x_1+4, 2x_2+3)$		

Table 5.3. The rectangles that describe the 2D-OPP's p and q from **Figure 5.3** under their respective associated grids.

Finally, and before to proceed to our nD-EVM's study, we clarify that the lattices and point grids we have defined have the objective of describing nD-OPP's as union of disjoint nD hyper-boxes in such way that by selecting a vertex, in any of these hyper-boxes, we have that such vertex is surrounded up to 2^n hyper-boxes. In this way, we can perform, as seen in previous chapter, geometrical and/or topological local analysis over such vertices. In **Section 5.6.1** we will describe the partition of nD-OPP's in disjoint nD hyper-boxes, defined by [Aguilera98], which is more appropriate for performing Regularized Boolean operations than our partitions.

5.2. Brinks and Extreme Vertices in the nD-OPP's

Definition 5.5: *A brink or extended edge is the maximal uninterrupted segment, built out of a sequence of collinear and contiguous* ***odd edges*** *of an nD-OPP*[1].

Property 5.1: *Even edges of an nD-OPP do not belong to brinks.*

Property 5.2: *Every odd edge belongs to brinks, whereas every brink consists of m edges, m ≥ 1, and contains m+1 vertices. Two of these vertices are at either extreme of the brink and the remaining m-1 are interior vertices.*

Definition 5.6: *Let p be an nD-OPP. A kD extended hypervolume of p, 1 < k < n, denoted by $\phi(p)$, is the maximal set of kD cells of p that lies in a kD space, such that a kD cell e_0 belongs to a kD extended hypervolume if and only if e_0 belongs to an (n-1)D cell present in $\partial(p)$, i.e.*

$$\left(e_0 \in \phi(p)\right) \Leftrightarrow \left(\exists c,\ c \text{ belongs to } \partial(p)\right)\left(e_0\left([0,1]^k\right) \subseteq c\left([0,1]^{n-1}\right)\right)$$

Definition 5.7: *Let p be an nD-OPP. V(p) will denote to set of vertices of p.*

Definition 5.8: *We will call Extreme Vertices of an nD-OPP p to the ending vertices of all the brinks in p. EV(p) will denote to the set of Extreme Vertices of p.*

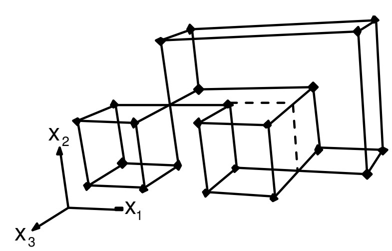


Figure 5.5. Example of a 3D-OPP *p* and its set of Extreme Vertices
(Continuous lines indicate odd edges while the dotted lines indicate even edges).

Figure 5.5 shows an example of a 3D-OPP and its set of Extreme Vertices. Consider the construction of a 4D-OPP as the union of several 4D-OPP's in the following way (**Figure 5.6.1**):

- We will have a 4D "*L-shaped*" polytope *a* in **Figure 5.6.1** (See **Appendix H**), and
- Three four-dimensional hyper-boxes *b*, *c* and *d*.
- The polytope *a* will share a vertex with hyper-box *c* and a face with hyper-box *b*, hence, all the edges in the shared face are characterized as even edges.
- The hyper-box *b* will share an edge with hyper-box *d*. Such edge for instance is an even edge.
- The remaining edges in the final polytope are characterized as odd edges.

See the final 4D-OPP in the **Figure 5.6.2**. The **Figure 5.6.3** shows its set of extreme vertices.

[1] The previous definition of brink, given in [Aguilera98], considered it as the maximal uninterrupted segment, build out of a sequence of collinear and contiguous manifold edges of a kD-OPP, k = 1, 2, 3. **Definition 5.5** is consistent with this previous one because we have commented in **Chapter 4** that in 1D, 2D and 3D spaces a manifold edge is equivalent to an odd edge.

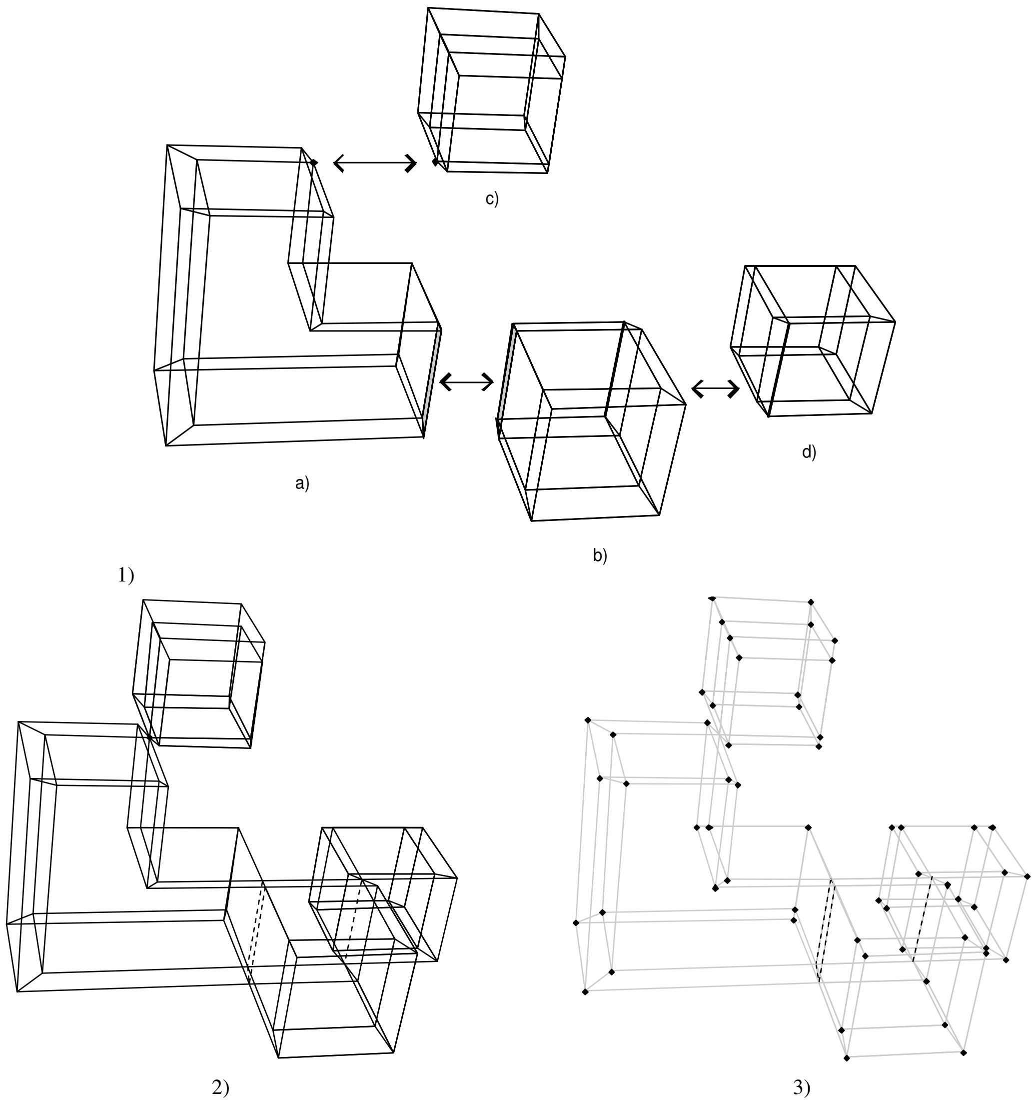


Figure 5.6. 1) The construction of a 4D-OPP by the union of several 4D-OPP's (a 4D "*L-shaped*" polytope and three hyper-boxes). 2) The wireframe model of the final polytope. b) Its set of Extreme Vertices (Continuous lines indicate odd edges while the dotted lines indicate even edges).

Property 5.3: *Let p be an nD-OPP then* $EV(p) \subseteq V(p)$.

The brinks in an nD-OPP p can be classified according to the main axis to which they are parallel. Since the extreme vertices mark the end of brinks in the n orthogonal directions, is that any of the n possible sets of brinks parallel to X_i-axis, $1 \leq i \leq n$, (see **Figures 5.7 and 5.8**), as it will be proved in **Corollary 5.4**, produce to the same set EV(p).

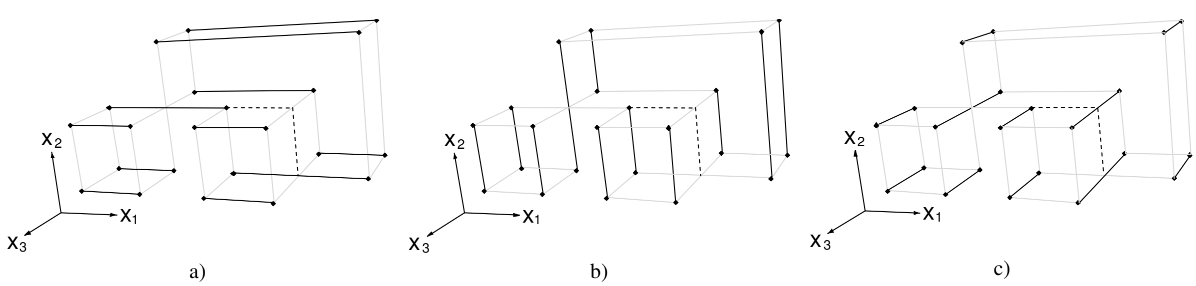


Figure 5.7. The brinks in a 3D-OPP (the OPP presented in **Figure 5.5**). a) The brinks parallel to X_1-axis, b) the brinks parallel to X_2-axis, c) the brinks parallel to X_3-axis (Continuous lines indicate odd edges while the dotted lines indicate even edges).

For example, in **Figure 5.8** are respectively shown:

- The parallel brinks to X_1-axis (**5.8.a**);
- The parallel brinks to X_2-axis (**5.8.b**);
- The parallel brinks to X_3-axis (**5.8.c**);
- The parallel brinks to X_4-axis (**5.8.d**);

From the 4D-OPP presented in **Figure 5.6.2**.

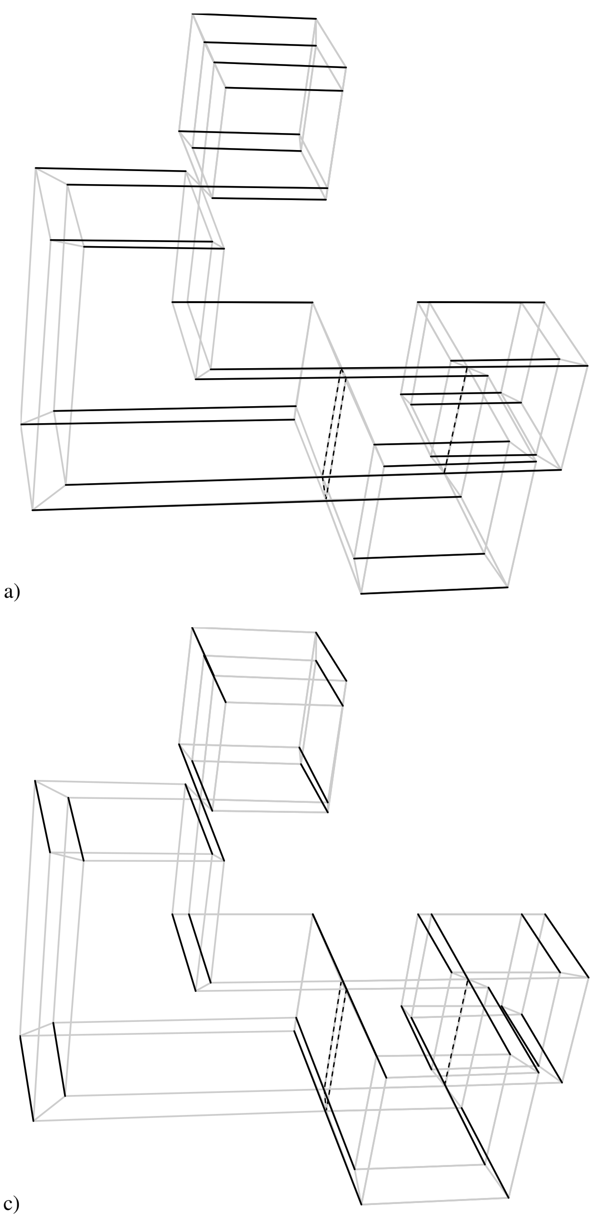


Figure 5.8. A 4D-OPP (from **Figure 5.6.2**) and its brinks parallel to X_1 (a), X_2 (b), X_3 (c) and X_4 (d) axes (Continuous lines indicate odd edges while the dotted lines indicate even edges).

Definition 5.9: *Let p be an nD-OPP.* $\underline{EV_i(p)}$ *will denote to the set of ending or extreme vertices of the brinks of p which are parallel to* x_i*-axis,* $1 \le i \le n$*.*

The Parallel Projection of an nD polytope onto an (n-1)D hyperplane, or in other words, the ***nD – (n-1)D Parallel Projection*** consists on just removing the j-th coordinate, whose corresponding axis is X_j, from the nD polytope's points [Aguilera02b]. In this work we will require a tool to project certain elements on the boundary of an nD-OPP under the idea behind nD – (n-1)D parallel projection. We formalize that idea with the following

Definition 5.10: *We define the* <u>*Projection Operator*</u> *for (n-1)D cells, main edges, points and set of points respectively as follows:*

- *Let* $c(I^n_{(i,\alpha)}(x)) = (x_1,..,x_n)$ *be an (n-1)D cell embedded in the nD space.* $\pi_j\left(c(I^n_{(i,\alpha)}(x))\right)$ *will denote the projection of the cell* $c(I^n_{(i,\alpha)}(x))$ *onto an (n-1)D space embedded in nD space whose supporting hyperplane is perpendicular to* X_j*-axis:*

$$\pi_j\left(c(I^n_{(i,\alpha)}(x))\right) = (x_1,...,\hat{x}_j,...,x_n)$$

- *Let* $c^{(1,i,\beta)} = (0,...,0,x_i,0,...,0)$ *be a main edge. The projection of such edge in the (n-1)D space, denoted by* $\pi_j\left(c^{(1,i,\beta)}\right)$*, is given by:*

$$\pi_j\left(c^{(1,i,\beta)}\right) = (\underbrace{0,...,\hat{0}}_{j},x_i,0,...,0)$$

- *Let* $v = (x_1,...,x_n)$ *a point in* $\mathbb{R}^n$*. The projection of that point in the (n-1)D space, denoted by* $\pi_j(v)$*, is given by:*

$$\pi_j(v) = (x_1,...,\hat{x}_j,...,x_n)$$

- *Let Q be a set of points in* $\mathbb{R}^n$*. We define the projection of the points in Q, denoted by* $\pi_j(Q)$*, as the set of points in* $\mathbb{R}^{n-1}$ *such that*

$$\pi_j(Q) = \left\{p \in \mathbb{R}^{n-1} : p = \pi_j(x),\ \ x \in Q \subset \mathbb{R}^n\right\}$$

In all the cases $\hat{x}_j$ *is the coordinate corresponding to* X_j*-axis to be suppressed.*

It should be noted that a main edge $c^{(1,i,\beta)}$ in nD space always has as projection, $\pi_j\left(c^{(1,i,\beta)}\right)$, a main edge in (n-1)D space except when $j = i$. In this last case the origin of (n-1)D space is obtained, i.e., $\pi_i\left(c^{(1,i,\beta)}\right) = (0,...,0)$.

Figure 5.9 shows the application of projection operator over some elements on the boundary of a 3D hyper-box. $\pi_1(c_1)$ projects face c_1 onto the plane X_2X_3 which is perpendicular to X_1-axis. In fact, the projection of face c_3 , $\pi_1(c_3)$, coincides with $\pi_1(c_1)$. These same faces projected onto plane X_1X_3 collapse in segments ($\pi_2(c_1)$ and $\pi_2(c_3)$). Let Q be set of vertices in that cube. Hence, by projecting Q onto plane X_1X_2, $\pi_3(Q)$, the collinear vertices in direction of axis X_3 have the same projection.

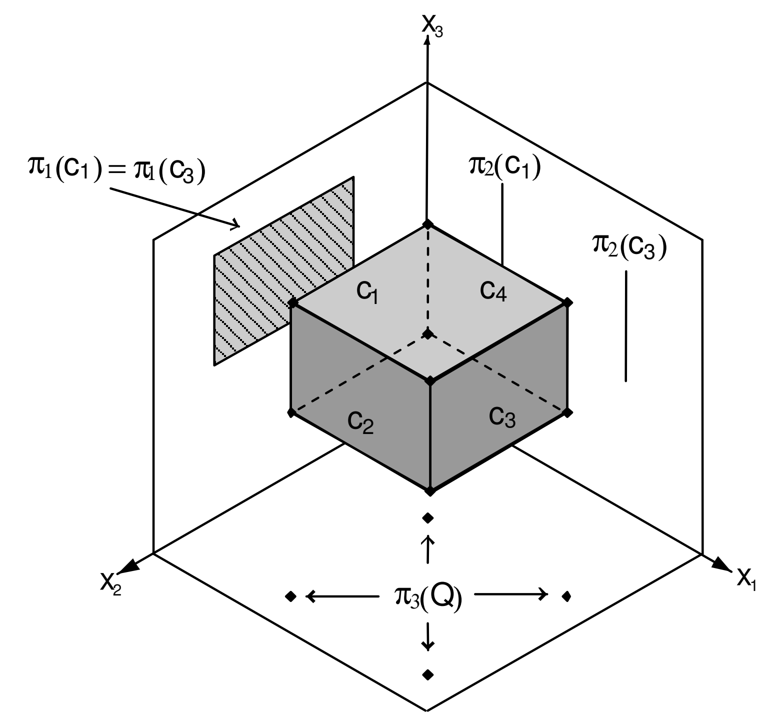


Figure 5.9. Projecting certain elements on the boundary of a 3D cube onto the 3D space's main planes (See text for details).

5.3. Local Analysis Over Extreme Vertices in the nD-OPP's

Theorem 5.1: *A vertex of an nD-OPP p, $n \geq 1$, when is locally described by a set of surrounding nD hyper-boxes, is an extreme vertex if and only if it is surrounded by an odd number of such nD hyper-boxes.*
Proof:
$\Rightarrow$)
Let v_0 be an extreme vertex in p. By **Definition 5.8** v_0 is one of the ending vertices of a brink. Therefore, it has exactly one incident odd edge which lies in one of the coordinate axes of the local nD space around v_0. Then, by **Lemma 4.1**, the number of nD hyper-boxes incident to v_0 is odd.
$\Leftarrow$)
Let v_0 a vertex of p and let c be the combination of nD hyper-boxes of p incident to v_0. Then, by **Theorem 4.2**, there are n linearly independent odd edges incident to v_0. Hence, v_0 is the ending vertex of n brinks parallel to each one of the coordinate axes of the local nD space around v_0. Therefore, v_0 is an extreme vertex. ◾

Corollary 5.1: *A vertex of an nD-OPP p, $n \geq 1$, when is locally described by a set of surrounding nD hyper-boxes, is an non-extreme vertex or a non-valid vertex if and only if it is surrounded by an even number of such nD hyper-boxes.*
Proof:
The proposition is the counterreciprocal of **Theorem 5.1** ($p \Leftrightarrow q \equiv \neg p \Leftrightarrow \neg q$). ◾

Theorem 5.2: *Any extreme vertex of an nD-OPP, $n \geq 1$, when is locally described by a set of surrounding nD hyper-boxes, has exactly n incident linearly independent odd edges.*
Proof:
Let v_0 an extreme vertex of p and let c be the combination of nD hyper-boxes of p incident to v_0. By **Theorem 5.1** we have that the number of nD hyper-boxes incident to v_0 is odd. Then, by **Theorem 4.2**, there are n linearly independent odd edges incident to v_0. ◾

Let c be a combination of nD hyper-boxes, $n > 1$, and consider the (n-1)D cells in $\partial(c)$ embedded in each one of the main (n-1)D hyperplanes. By applying the projection operator to such embedded cells by suppressing the x_j coordinate, which corresponds to the axis which is perpendicular to its corresponding (n-1)D hyperplane, we get an (n-1)D combination of (n-1)D hyper-boxes.

Lemma 5.1: *Let c be a combination of nD hyper-boxes, $n > 1$. If c describes an non-extreme vertex then in each one of the n main (n-1)D hyperplanes, the (n-1)D-OPP's composed by the (n-1)D cells in $\partial(c)$ embedded in such hyperplanes describe non-extreme vertices.*
Proof:
If c describes a non-extreme vertex then, by **Corollary 5.1**, $\Gamma(c)$ is an even number. By **Corollary 4.1**, in each axis X_i, $1 \leq i \leq n$, there is a pair of collinear odd edges or there is a pair of collinear even edges. By **Corollary 4.2**, the number of (n-1)D hyper-boxes in $\partial(c)$ embedded in each one of the n main hyperplanes is even. By applying the projection operator π_i over the (n-1)D hyper-boxes embedded in the hyperplane perpendicular to X_i-axis we obtain an (n-1)D-OPP, namely an (n-1)D combination, whose (n-1)D hyper-boxes describe, by **Corollary 5.1**, a non extreme vertex in (n-1)D space. ◾

Theorem 5.3: *Let c be a combination of nD hyper-boxes, $n > 1$. Combination c describes an extreme vertex if and only if in each one of the n main (n-1)D hyperplanes, the (n-1)D-OPP's composed by the (n-1)D cells in $\partial(c)$ embedded in such hyperplanes describe extreme vertices.*
Proof:
$\Rightarrow$)
If combination c describes an extreme vertex then, by **Theorem 5.2**, it has exactly n incident linearly independent odd edges. Consider an odd edge on X_i-axis, $1 \leq i \leq n$. By **Theorem 4.4** the number of (n-1)D hyper-boxes in $\partial(c)$ which are perpendicular to the odd edge on X_i-axis is odd. By applying the projection operator π_i over such (n-1)D hyper-boxes we obtain an (n-1)D-OPP, namely an (n-1)D combination, whose (n-1)D hyper-boxes describe, by **Theorem 5.1**, an extreme vertex in (n-1)D space.

By applying the above procedure to the remaining n-1 axes, in which lies exactly one odd edge, we will get the same result. Thus, we will have n extreme vertices associated to the (n-1)D-OPP's embedded in all the main hyperplanes of combination c.

$\Leftarrow$)

The reciprocal is the counterreciprocal of **Lemma 5.1** ($p \Rightarrow q \equiv \neg q \Rightarrow \neg p$). ◪

Table 5.4 shows three examples of 3D combinations of boxes that describe Extreme Vertices because they have an odd number of boxes. It can be observed in the three cases that by considering only such faces incident to the origin and included in the boundary of the combination, by grouping them according to its supporting main plane (X_1X_2, X_2X_3 or X_1X_3), their corresponding projections generate 2D combinations of rectangles which in turn also describe Extreme Vertices because they have an odd number of rectangles.

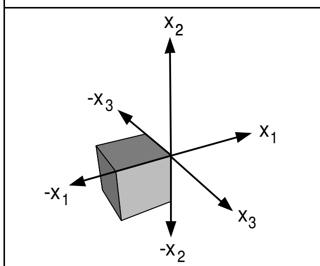
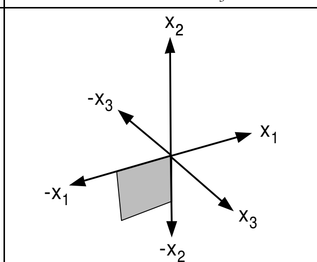
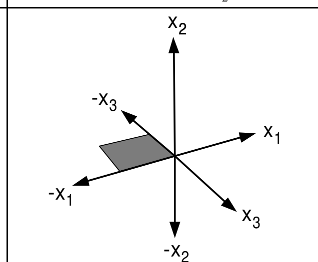
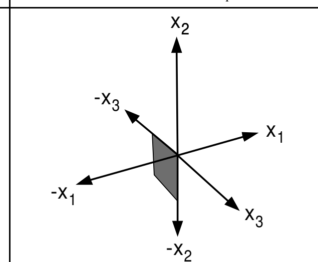
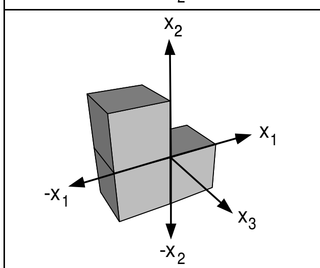
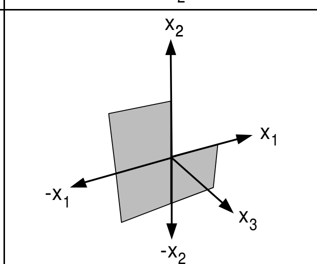
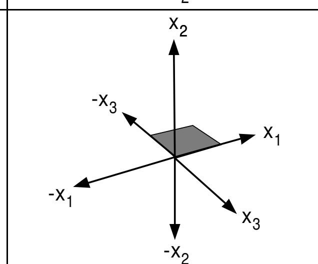
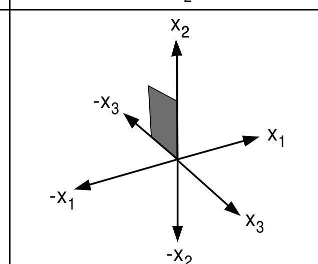
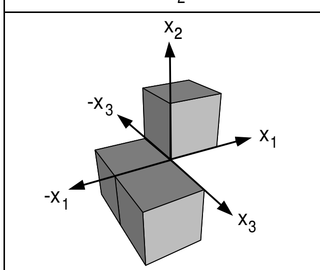
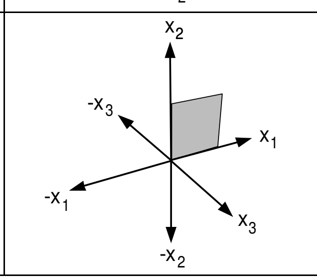
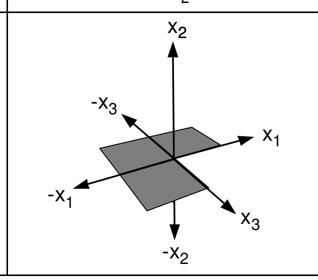
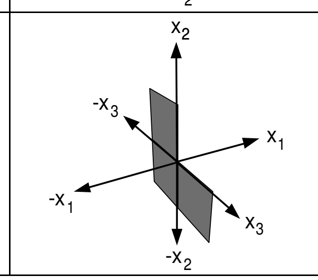
3D combination	**Projection onto X_1X_2 plane** (π_3)	**Projection onto X_1X_3 plane** (π_2)	**Projection onto X_2X_3 plane** (π_1)
			
			
			

Table 5.4. Three 3D combinations of boxes that describe Extreme Vertices. The projections of those faces included in the boundary of the combination and incident to the Extreme Vertex also describe Extreme Vertices in 2D space.

Corollary 5.2: *Let c be a combination of nD hyper-boxes, n > 1. Combination c describes a non-extreme vertex if and only if in each one of the n main (n-1)D hyperplanes, the (n-1)D-OPP's composed by the (n-1)D cells in $\partial(c)$ embedded in such hyperplanes describe non-extreme vertices.*

Proof:

The proposition is the counterreciprocal of **Theorem 5.3** ($p \Leftrightarrow q \equiv \neg p \Leftrightarrow \neg q$). ◪

Theorem 5.4: *Let c be a combination of nD hyper-boxes which describes an extreme vertex, n > 1. The number of (n-1)D cells in ∂(c) incident to the extreme vertex is:*

- *Odd if n is odd, or*
- *Even if n is even*

Proof:
By **Theorem 5.3**, because combination c describes an extreme vertex then in each one of the (n-1)D hyperplanes there is an odd number of (n-1)D cells which are included in ∂(c). In nD space there are n main (n-1)D hyperplanes which pass through the origin. Hence, if n is an even number then the total sum of (n-1)D cells in ∂(c) which are incident to the origin is even. In the other hand, if n is an odd number then the total sum of (n-1)D cells in ∂(c) which are incident to the origin is odd. ◻

Theorem 5.5: *Let c be a combination of nD hyper-boxes which describes a non-extreme vertex, n > 1. The number of (n-1)D cells in ∂(c) incident to the non-extreme vertex is even.*
Proof:
By **Corollary 5.2**, because combination c describes a non-extreme vertex then in each one of the (n-1)D hyperplanes there is an even number of (n-1)D cells which are included in ∂(c). In nD space there are n main (n-1)D hyperplanes which pass through the origin. Hence, the total sum of (n-1)D cells in ∂(c) which are incident to the origin is even. ◻

Appendix E shows the possible characterizations of Extreme Vertices in 2D, 3D and 4D-OPP's according to **Theorems 5.1** to **5.4**. Moreover, the possible characterizations of non-extreme vertices in these same polytopes are also shown (which are given according to **Corollaries 5.1** to **5.2**).

Lemma 5.2: *Let c be a combination of nD hyper-boxes, n > 1, and let e_0 be an edge on x_i-axis, $1 \leq i \leq n$. If e_0 is an even edge then in each one of the n main (n-1)D hyperplanes, where e_0 is embedded, $\pi_j(e_0)$, for all $j \neq i$, is an even edge of the (n-1)D-OPP's composed by the (n-1)D cells in ∂(c) embedded in such hyperplanes.*
Proof:
If e_0 is an even edge then, by **Corollary 4.3,** the number of coplanar (n-1)D cells in ∂(c) which are incident to e_0 is even. By applying the projection operator π_j, $j \neq i$, over all the (n-1)D hyper-boxes embedded in the hyperplane perpendicular to X_j-axis we obtain an (n-1)D-OPP, namely an (n-1)D combination, in which an even number of its (n-1)D hyper-boxes are incident to the main edge $\pi_j(e_0)$. Hence, $\pi_j(e_0)$ is an even edge for all $j \neq i$. ◻

Theorem 5.6: *Let c be a combination of nD hyper-boxes, n > 1, and let e_0 be an edge on x_i-axis, $1 \leq i \leq n$. Edge e_0 is an odd edge if and only if in each one of the n main (n-1)D hyperplanes, where e_0 is embedded, $\pi_j(e_0)$, for all $j \neq i$, is an odd edge of the (n-1)D-OPP's composed by the (n-1)D cells in ∂(c) embedded in such hyperplanes.*
Proof:
⇒)
If e_0 is an odd edge then, by **Theorem 4.6,** the number of coplanar (n-1)D cells in ∂(c) which are incident to e_0 is odd. By applying the projection operator π_j, $j \neq i$, over all the (n-1)D hyper-boxes embedded in the hyperplane perpendicular to X_j-axis we obtain an (n-1)D-OPP, namely an (n-1)D combination, in which an odd number of its (n-1)D hyper-boxes are incident to the main edge $\pi_j(e_0)$. Hence, $\pi_j(e_0)$ is an odd edge for all $j \neq i$.
⇐)
The reciprocal is the counterreciprocal of **Lemma 5.2** ($p \Rightarrow q \equiv \neg q \Rightarrow \neg p$). ◻

Corollary 5.3: *Let c be a combination of nD hyper-boxes, n > 1, and let e_0 be an edge on x_i-axis, $1 \leq i \leq n$. Edge e_0 is an even edge if and only if in each one of the n main (n-1)D hyperplanes, where e_0 is embedded, $\pi_j(e_0)$, for all $j \neq i$, is an even edge of the (n-1)D-OPP's composed by the (n-1)D cells in ∂(c) embedded in such hyperplanes.*
Proof:
The proposition is the counterreciprocal of **Theorem 5.6** ($p \Leftrightarrow q \equiv \neg p \Leftrightarrow \neg q$). ◻

Theorem 5.7 (Characterization of Extreme Vertices according to their incident odd and even edges): *Let v_0 be an Extreme Vertex in a combination of nD hyper-boxes. Vertex v_0 has exactly n incident linearly independent odd edges and n incident linearly independent even edges.*

Proof:

By **Theorem 5.2**, v_0 has n incident odd edges embedded in each one of the main axes of the local coordinate system described by v_0. Because there is exactly one odd edge e_0 embedded in x_i^+ or x_i^-, $1 \le i \le n$, then the number of nD hyper-boxes incident to e_0 is odd. Hence, there is an even edge which is collinear to e_0. Therefore, for all $i \in \{1,...,n\}$, X_i-axis has one odd edge and one even edge incident to v_0 which yields to n incident even edges incident to v_0. ◪

Theorem 5.8 (Characterization of Non-Extreme Vertices according to their incident odd and even edges): *There are n+1 types of non-extreme vertices that can be present in a combination of nD hyper-boxes, that is, a non-extreme vertex has (n-i) pairs of incident collinear odd edges and i pairs of incident collinear even edges, for* $i \in \{0,1,...,n\}$.

Proof:

By **Corollary 5.1**, a non-extreme vertex v_0 has an even number of incident hyper-boxes. By **Corollary 4.1**, in each one of the main axes, of the local coordinate system described by v_0, there are pairs of collinear odd edges or pairs of collinear even edges incident to v_0. The n+1 types arise by considering all possible combinations of presence of pairs of collinear odd edges or pairs of collinear even edges. ◪

At this point is important to consider that the notion of non-extreme vertex with two incident even edges is not present only in 1D space, because even edges or non-valid edges are not possible in such space. The **Table 5.6** shows the extreme and non-extreme vertices present in the 1D, 2D, 3D and 4D-OPP's characterized according to their incident odd and even edges.

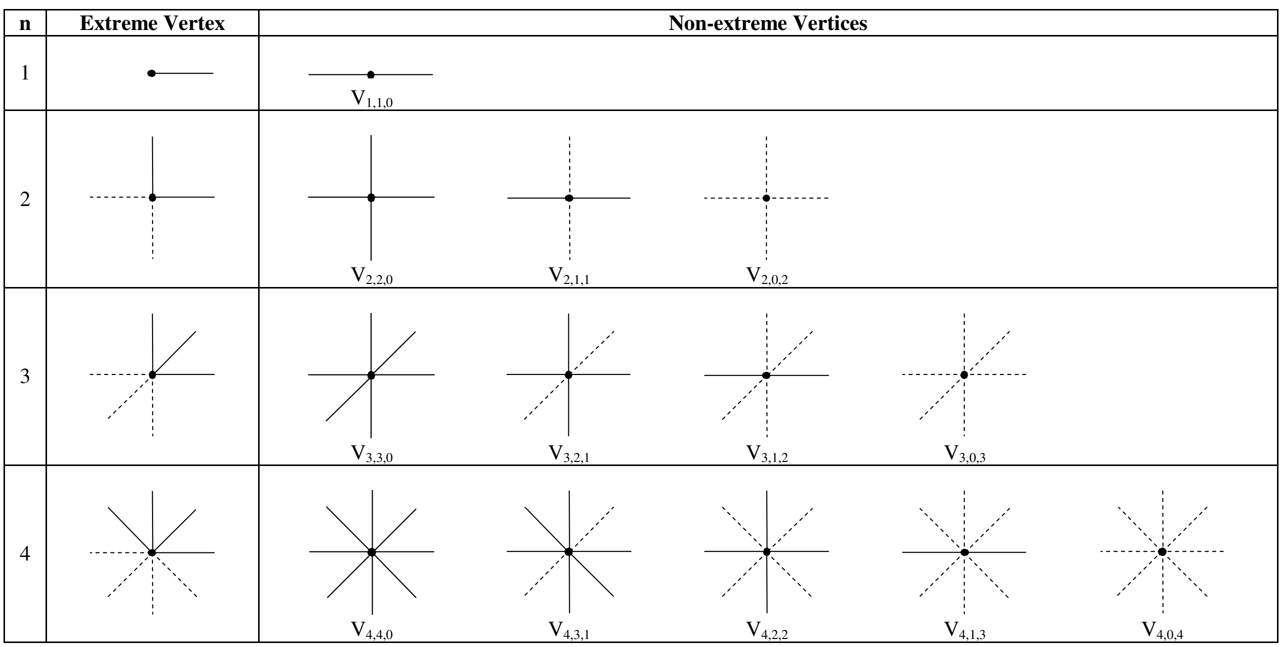


Table 5.6. Characterization of extreme and non-extreme vertices in the nD-OPP's, $n \in \{1, 2, 3, 4\}$, according to incident odd and even edges (——— : Odd edge, ········ : Even edge).

The **Appendix F** shows the correspondences between the characterizations of Extreme and non-extreme Vertices according to **Theorems 5.7** and **5.8**, with the characterizations of Extreme and non-extreme vertices according to the procedures described originally in [Aguilera98] (in the 1D, 2D and 3D-OPP's) and [Pérez-Aguila03d] (in the 4D-OPP's).

5.4. Global Analysis Over the nD-OPP's and the Extreme Vertices Model

Definition 5.11: *Let p be an nD-OPP. Let* $\underline{VX_i = \{a_1, a_2, ..., a_{nx_i}\}}$ *be the set of different values such that $a_j < a_{j+1}$ and a_j is the coordinate associated to X_i-axis of some extreme vertex* $v_k \in EV(p)$ *where $1 \le j \le nx_i$ and $1 \le k \le Card(EV(p))$.*

Definition 5.12: *Let p be an nD-OPP.* $V_{n,i}$ *will refer to a non-extreme vertex of p with i pairs of incident collinear odd edges and n-i pairs of incident collinear even edges.* $V_{n,i}(p)$ *will refer to the set of non-extreme vertices of type* $V_{n,i}$ *of p.*

Property 5.4: *Let p be an nD-OPP and let V(p) the set of all the vertices of p. V(p) is described as the union of the following disjoint subsets*

$$V(p) = EV(p) \cup V_{n,n}(p) \cup V_{n,(n-1)}(p) \cup \ldots \cup V_{n,1}(p) \cup V_{n,0}(p)$$

That is

$$V(p) = EV(p) \cup \bigcup_{i=0}^{n} V_{n,(n-i)}(p)$$

Theorem 5.9 (Obtaining non-extreme vertices' coordinates from Extreme Vertices): *Let p be an nD-OPP. Let v = {a_0, …, a_n} be an non-extreme vertex in p, i.e.,*

$$v \in \bigcup_{i=0}^{n} V_{n,(n-i)}(p)$$

Then $a_1 \in VX_1$, $a_2 \in VX_2$, …, $a_n \in VX_n$.

Proof:

The pair of Extreme Vertices of a brink, parallel to X_i-axis, share the values of (n-1) coordinates, where the coordinate corresponding to X_i-axis is that in where they differ. Hence the coordinates of *v* can be obtained directly if it is a vertex of type $V_{n,n}$, $V_{n,(n-1)}$, …, or $V_{n,2}$, i.e., if *v* is in the interior of at least two intersecting perpendicular brinks. In these cases the coordinates of *v* can be obtained from the ending vertices' coordinates of those brinks, then $a_1 \in VX_1$, $a_2 \in VX_2$, …, and $a_n \in VX_n$. If *v* is a vertex of type $V_{n,1}$, that is, *v* is in the interior of only one brink, then (n-1) of its n coordinates can be known directly. The remaining coordinate, as well as all n coordinates of a vertex of type $V_{n,0}$, can be obtained using the fact that the other end of an even edge can be either a vertex with known coordinates (a vertex in $EV(p) \cup V_{n,n}(p) \cup V_{n,(n-1)}(p) \cup \ldots \cup V_{n,2}(p)$), or again, a vertex of type $V_{n,1}$ or $V_{n,0}$. Therefore, in the last case, another even edge of the new vertex can be used to repeat this procedure until a vertex with known coordinates is reached. $\square$

Theorem 5.10: *Let p be an nD-OPP with its associated sets $EV_1(p)$, $EV_2(p)$, …, $EV_{n-1}(p)$, $EV_n(p)$. Then*

$$EV_1(p) = EV_2(p) = \ldots = EV_{n-1}(p) = EV_n(p)$$

Proof:

Let $EV_i(p)$ and $EV_j(p)$, $i \neq j$, $1 \leq i, j \leq n$, be any two sets of extreme vertices of the brinks of p which are parallel to x_i and x_j axes respectively. We will show that $EV_i(p) = EV_j(p)$ by proving the double inclusion.

$\subseteq$)

Let v_0 be any extreme vertex in $EV_i(p)$. Then, by **Theorem 5.2**, n linearly independent odd edges are incident to v_0. One of these odd edges is parallel to x_j-axis. Hence, $v_0 \in EV_j(p)$ $\therefore$ $EV_i(p) \subseteq EV_j(p)$

$\supseteq$)

By using the previous reasoning we conclude that $EV_i(p) \supseteq EV_j(p)$ because any extreme vertex in $EV_j(p)$ is in $EV_i(p)$ $\therefore$ $EV_i(p) \supseteq EV_j(p)$

$\therefore$ $EV_i(p) = EV_j(p)$ $\square$

Corollary 5.4: *Let p be an nD-OPP. Then $EV(p) = EV_i(p)$, $1 \leq i \leq n$.*

Proof:

$\subseteq$)

Any extreme vertex in EV(p) is included in $EV_i(p)$ because one of its incident n odd edges is parallel to x_i-axis $\therefore$ $EV(p) \subseteq EV_i(p)$

$\supseteq$)

Any extreme vertex in $EV_i(p)$ is in EV(p) because it has n-1 incident odd edges besides of its incident odd edge which is parallel to x_i-axis $\therefore$ $EV(p) \supseteq EV_j(p)$

$\therefore$ $EV(p) = EV_j(p)$ $\square$

Lemma 5.3: *Let p be an nD-OPP. Card(EV_i(p)) is an even number, $1 \leq i \leq n$.*
Proof:
By definition, EV_i(p) is composed by all the extreme vertices of the brinks of p which are parallel to x_i-axis. By **Property 5.2** every brink has two extreme (or ending) vertices. Let m_i be the number of brinks of p which are parallel to x_i-axis. Then Card(EV_i(p)) = $2m_i$ is an even number. □

Let Q be a finite set of points in $\mathbb{R}^3$. In [Aguilera98] was defined the ABC-sorted set of Q as the set resulting from sorting Q according to coordinate A, then to coordinate B, and then to coordinate C. For instance, a set Q can be ABC-sorted is six different ways: $X_1X_2X_3$, $X_1X_3X_2$, $X_2X_1X_3$, $X_2X_3X_1$, $X_3X_1X_2$ and $X_3X_2X_1$. Now, let p be a 3D-OPP. According to [Aguilera98] the Extreme Vertices Model of p, EVM(p), denotes to the ABC-sorted set of the extreme vertices of p. Then EVM(p) = EV(p) except by the fact that EV(p) is not necessarily sorted. In this work we will assume that the coordinates of extreme vertices in the Extreme Vertices Model of an nD-OPP p, EVM_n(p) are sorted according to coordinate X_1, then to coordinate X_2, and so on until coordinate X_n. That is, we are considering the only ordering $X_1 \ldots X_i \ldots X_n$ such that $i-1 < i$, $1 < i \leq n$.

Definition 5.13: *Let p be an nD-OPP. We will define the <u>Extreme Vertices Model</u> of p, denoted by <u>EVM_n(p)</u>, as the model as only stores to all the extreme vertices of p.*

Theorem 5.11: *Let p be an nD-OPP.*
1) *Card(EV(p)) is an even number.*
2) *Card(EVM_n(p)) is an even number.*

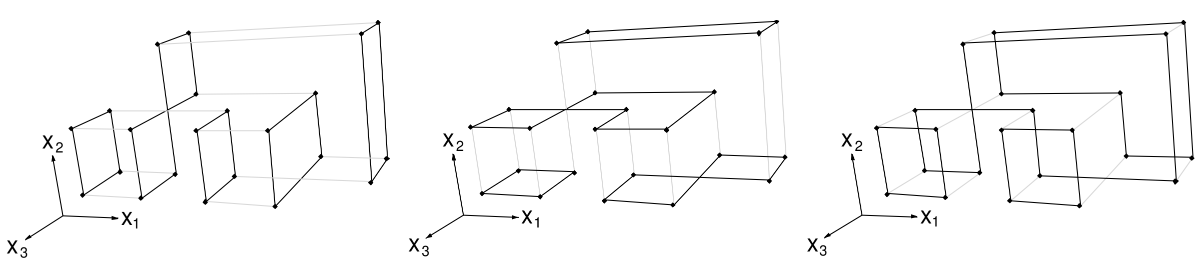
Proof:
1) By **Corollary 5.4**, EV(p) = EV_i(p), $1 \leq i \leq n$ and by **Lemma 5.3** Card(EV(p)) = Card(EV_i(p)) is an even number.
2) By definition EVM_n(p) = EV(p) and by 1) Card(EVM_n(p)) = Card(EV(p)) is an even number. □

5.4.1. Relating the nD-EVM with the (n-1)D-EVM of an (n-1)D-Couplet

Definition 5.14: *Consider an nD-OPP p:*
- *Let $\underline{np_i}$ be the number of distinct coordinates present in the vertices of p along X_i-axis, $1 \leq i \leq n$.*
- *Let $\underline{\Phi_k^i(p)}$ be the k-th (n-1)D extended hypervolume of p which is perpendicular to X_i-axis, $1 \leq k \leq np_i$.*
- *Let $\underline{H\left(\Phi_k^i(p)\right)}$ be the (n-1)D hyperplane where $\Phi_k^i(p)$ lies.*
- *Let $\underline{EV_k^i(p)} \subset EVM_n(p)$ be the set of extreme vertices embedded in $H\left(\Phi_k^i(p)\right)$, $1 \leq i \leq n$, $1 \leq k \leq np_i$.*

In **Chapter 4** we defined an (n-1)D-couplet as a set of (n-1)D-coupled cells, i.e., (n-1)D cells embedded in a same (n-1)D hyperplane. For the sake of the simplicity in the terminology used in this work, starting from this section, the set of cells in an (n-1)D extended hypervolume will be referred as an (n-1)D-couplet or just a couplet.

The **Figure 5.10** shows as an example the sequences of 2D-couplets, i.e. 2D extended hypervolumes, of a 3D-OPP. Such sequences are categorized according to the 3D space main axis which is perpendicular to those 2D-couplets.



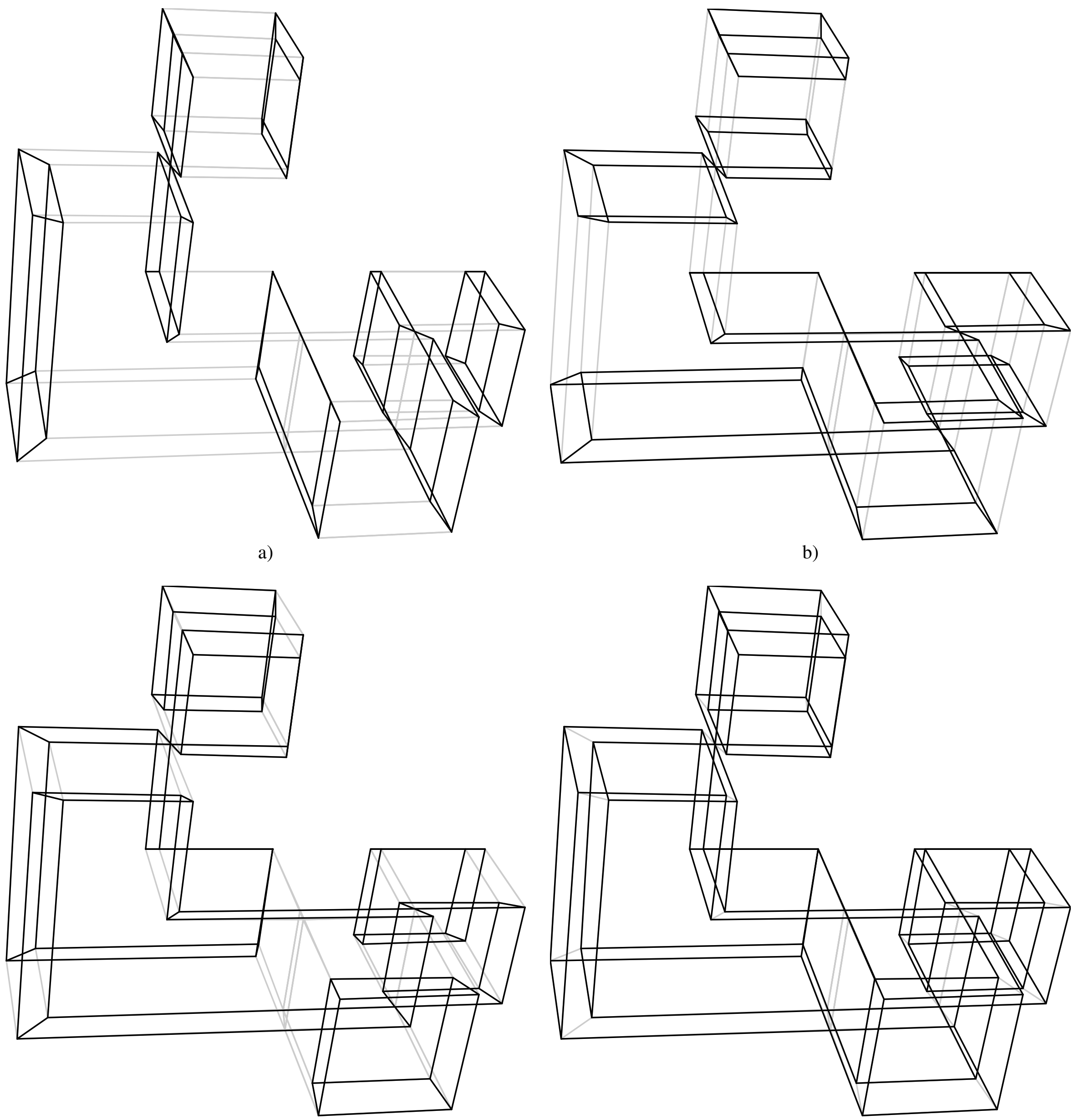
a) b) c)

Figure 5.10. The sequences of 2D-couplets in a 3D-OPP (the OPP presented in **Figure 5.5**). a) The 2D-couplets perpendicular to X_1-axis. b) The 2D-couplets perpendicular to X_2-axis. c) The 2D-couplets perpendicular to X_3-axis.

For example, in **Figure 5.11** are respectively shown (couplets):

- The Φ's perpendicular to X_1-axis (**5.11.a**);
- The Φ's perpendicular to X_2-axis (**5.11.b**);
- The Φ's perpendicular to X_3-axis (**5.11.c**);
- The Φ's perpendicular to X_4-axis (**5.11.d**);

From the 4D-OPP presented in **Figure 5.6.2**.



a) b)

c) d)

Figure 5.11. A 4D-OPP (from **Figure 5.6.2**) and its couplets perpendicular to X_1 (a), X_2 (b), X_3 (c) and X_4 (d) axes.

Lemma 5.4: *Let p be an nD-OPP. Let v_0 a non-extreme vertex of p embedded in the supporting hyperplane of* $\Phi_k^j(p)$, $H\left(\Phi_k^i(p)\right)$, *which is perpendicular to X_j-axis. If* $v_0 \notin EV_k^j(p)$ *then* $\Pi_j(v_0) \notin EVM_{n-1}\left(\pi_j\left(\Phi_k^j(p)\right)\right)$, $1 \le j \le n$, $1 \le k \le np_j$.

Proof:
If v_0 is a non-extreme vertex of p then, by **Corollary 5.2**, $\Pi_i(v_0)$ is a non-extreme vertex of the (n-1)D-OPP's embedded in each one of the main hyperplanes that pass through v_0 seen as the origin of its local coordinate system, $1 \le i \le n$. When i = j we have the particular case when X_j-axis is perpendicular to $H\left(\Phi_k^j(p)\right)$. Therefore $\pi_j(v_0) \notin EVM_{n-1}\left(\pi_j\left(\Phi_k^j(p)\right)\right)$. ◪

Lemma 5.5: *Let p be an nD-OPP. The set of extreme vertices of the projection of an (n-1)D-couplet of p,* $\Phi_k^j(p)$*, is equal to the projection of the set of extreme vertices* $EV_k^j(p)$, i.e., $EVM_{n-1}(\pi_j(\Phi_k^j(p))) = \pi_j(EV_k^j(p))$, *$1 \le j \le n$, $1 \le k \le np_j$.*
Proof:
$\subseteq$)
Through the counterreciprocal of **Lemma 5.4** we have that if $\pi_j(v_0) \in EVM_{n-1}\left(\pi_j\left(\Phi_k^j(p)\right)\right)$ then $v_0 \in EV_k^j(p)$. Hence, $EVM_{n-1}(\pi_j(\Phi_k^j(p))) \subseteq \pi_j(EV_k^j(p))$.
$\supseteq$)
By **Theorem 5.3** each extreme vertex v_0 in $EV_k^j(p)$ is an extreme vertex $\pi_j(v_0)$ of $EVM_{n-1}(\pi_j(\Phi_k^j(p)))$. Therefore, $EVM_{n-1}(\pi_j(\Phi_k^j(p))) \supseteq \pi_j(EV_k^j(p))$.

$\therefore EVM_{n-1}(\pi_j(\Phi_k^j(p))) = \pi_j(EV_k^j(p))$ ◪

Theorem 5.12: *Let p be an nD-OPP. The set of extreme vertices of the projection of an (n-1)D-couplet of p,* $\Phi_k^j(p)$*, is a subset of* $\pi_j(EVM_n(p))$*, i.e.,* $EVM_{n-1}\left(\pi_j(\Phi_k^j(p))\right) \subseteq \pi_j(EVM_n(p))$*, where X_j-axis is perpendicular to the supporting hyperplane of* $\Phi_k^j(p)$*, $1 \le j \le n$, $1 \le k \le np_j$.*
Proof:
Let $EV_k^j(p)$ be the set of extreme vertices embedded in $H\left(\Phi_k^j(p)\right)$. Obviously $EV_k^j(p) \subseteq EVM_n(p)$ and $\pi_j(EV_k^j(p)) \subseteq \pi_j(EVM_n(p))$. By **Lemma 5.5** we have that $EVM_{n-1}(\pi_j(\Phi_k^j(p))) = \pi_j(EV_k^j(p))$, therefore $EVM_{n-1}(\pi_j(\Phi_k^j(p))) \subseteq \pi_j(EVM_n(p))$. ◪

Theorem 5.13: *Consider the X_j-axis, $1 \le j \le n$, and let p be an nD-OPP. The union of the EVMs corresponding to the projection of all the (n-1)D-couplets of p whose supporting hyperplane is perpendicular to X_j-axis is the projection of the EVM of p, i.e.,*

$$\pi_j\left(EVM_n(p)\right) = \bigcup_{k=1}^{np_j} EVM_{n-1}\left(\pi_j\left(\Phi_k^j\right)\right)$$

Proof:
$\subseteq$)
Let v_0 an extreme vertex in $EVM_n(p)$. Then there exists a value k', $1 \le k' \le np_j$, such that $v_0 \in EV_{k'}^i(p)$. By **Lemma 5.5**, if $\pi_j(v_0) \in \pi_j(EV_{k'}^j(p))$ then $\pi_j(v_0) \in EVM_{n-1}(\pi_j(\Phi_{k'}^j(p))) \subseteq \bigcup_{k=1}^{np_j} EVM_{n-1}\left(\pi_j\left(\Phi_k^j\right)\right)$. Hence $\pi_j\left(EVM_n(p)\right) \subseteq \bigcup_{k=1}^{np_j} EVM_{n-1}\left(\pi_j\left(\Phi_k^j\right)\right)$.
$\supseteq$)
Let $EVM_{n-1}(\pi_j(\Phi_{k'}^j(p))) \subseteq \bigcup_{k=1}^{np_j} EVM_{n-1}\left(\pi_j\left(\Phi_k^j\right)\right)$, $1 \le k' \le np_j$. By **Theorem 5.12**, $EVM_{n-1}\left(\pi_j(\Phi_{k'}^j(p))\right) \subseteq \pi_j(EVM_n(p))$.
Therefore $\bigcup_{k=1}^{np_j} EVM_{n-1}\left(\pi_j\left(\Phi_k^j\right)\right) \subseteq \pi_j\left(EVM_n(p)\right)$.

$\therefore \pi_j\left(EVM_n(p)\right) = \bigcup_{k=1}^{np_j} EVM_{n-1}\left(\pi_j\left(\Phi_k^j\right)\right)$ ◪

Lemma 5.6: *Let p be an nD-OPP. Let* $EV_k^j(p)$ *be the set of extreme vertices embedded in the (n-1)D hyperplane associated to the (n-1)D-couplet* $\Phi_k^j(p)$, *1 ≤ j ≤ n, 1 ≤ k ≤ np$_j$. Then,* $Card\left(EV_k^j(p)\right)$ *is even.*

Proof:
Those brinks of p which are embedded $H\left(\Phi_k^j(p)\right)$ have their two extreme vertices in $EV_k^j(p)$. Consider the brinks parallel to X_i-axis, i ≠ j. Let m_i be the number of those brinks. Hence, the number of extreme vertices in $EV_k^j(p)$ is given by $Card\left(EV_k^j(p)\right) = 2m_i$ which is an even number. ◪

Theorem 5.14: *The cardinality of the set of extreme vertices of the projection of an (n-1)D-couplet of p,* $\Phi_k^j(p)$, *is even, 1 ≤ j ≤ n, 1 ≤ k ≤ np$_j$.*
Proof:
Let $EV_k^j(p)$ be the set of extreme vertices embedded in the (n-1)D hyperplane associated to the (n-1)D-couplet $\Phi_k^j(p)$. By **Lemma 5.5** $\pi_j(EV_{k'}^j(p)) = EVM_{n-1}(\pi_j(\Phi_{k'}^j(p)))$, then we have that $Card\left(EVM_{n-1}(\pi_j(\Phi_{k'}^j(p)))\right) = Card(EV_{k'}^j(p))$ and by **Lemma 5.6**, $Card\left(EVM_{n-1}(\pi_j(\Phi_{k'}^j(p)))\right)$ is even. ◪

It should be noted that all the Extreme Vertices in $EV_k^i(p)$ have the same x_i coordinate. Hence, the sets $EV_k^i(p)$, for all k ∈ [1, np$_i$] induce a partition of the Extreme Vertices of an nD-OPP. We will proceed to formalize this observation through **Definition 5.15**, **Theorem 5.15** and **Property 5.5**.

Definition 5.15: *Let p be an nD-OPP. Let x_0 and y_0 be vertices in EVM$_n$(p). We define the relation* $\underline{R_{EV_k^i(p)}}$ *as*

$$x_0\ R_{EV_k^i(p)}\ y_0 \Leftrightarrow \left(x_0 \in EV_k^i(p)\right) \wedge \left(y_0 \in EV_k^i(p)\right)$$

Theorem 5.15: *The relation* $R_{EV_k^i(p)}$ *is an equivalence relation on the Extreme Vertices of an nD-OPP p.*

Proof:
Let x_0, y_0 and z_0 be Extreme Vertices in EVM$_n$(p). The following properties are satisfied:

- Reflexivity: $(\forall x_0 \in EVM_n(p))\left(x_0\ R_{EV_k^i(p)}\ x_0\right)$
- Symmetry: If $x_0\ R_{EV_k^i(p)}\ y_0 \Rightarrow \left(x_0 \in EV_k^i(p)\right) \wedge \left(y_0 \in EV_k^i(p)\right) \Rightarrow \left(y_0 \in EV_k^i(p)\right) \wedge \left(x_0 \in EV_k^i(p)\right) \Rightarrow y_0\ R_{EV_k^i(p)}\ x_0$
  $\therefore (\forall x_0, y_0 \in EVM_n(p))\left(x_0\ R_{EV_k^i(p)}\ y_0 \Rightarrow y_0\ R_{EV_k^i(p)}\ x_0\right)$
- Transitivity: If $\left(x_0\ R_{EV_k^i(p)}\ y_0\right) \wedge \left(y_0\ R_{EV_k^i(p)}\ z_0\right) \Rightarrow \left(\left(x_0 \in EV_k^i(p)\right) \wedge \left(y_0 \in EV_k^i(p)\right)\right) \wedge \left(\left(y_0 \in EV_k^i(p)\right) \wedge \left(z_0 \in EV_k^i(p)\right)\right)$
  $\Rightarrow \left(x_0 \in EV_k^i(p)\right) \wedge \left(z_0 \in EV_k^i(p)\right) \Rightarrow \left(x_0\ R_{EV_k^i(p)}\ z_0\right)$
  $\therefore (\forall x_0, y_0, z_0 \in EVM_n(p))\left(\left(x_0\ R_{EV_k^i(p)}\ y_0\right) \wedge \left(y_0\ R_{EV_k^i(p)}\ z_0\right) \Rightarrow x_0\ R_{EV_k^i(p)}\ z_0\right)$

$\therefore$ $R_{EV_k^i(p)}$ is an equivalence relation. ◪

Property 5.5: *Because* $EV_k^i(p)$ *are equivalence classes that partition the set of Extreme Vertices of an nD-OPP p we have:*

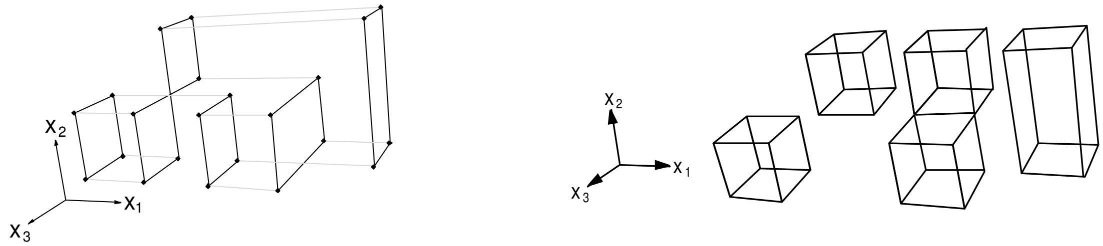
- $EV_k^i(p) \cap EV_{k'}^i(p) = \varnothing,\ k \neq k',\ 1 \leq k, k' \leq np_i$
- $EVM_n(p) = \bigcup_{k=1}^{np_i} EV_k^i(p)$

5.4.2. Sections and Slices of nD-OPP's

Definition 5.16: *A* <u>*Slice*</u> *is the region contained in an nD-OPP p between two consecutive couplets of p.* $\underline{Slice_k^i(p)}$ *will denote to the k-th slice of p which is bounded by* $\Phi_k^i(p)$ *and* $\Phi_{k+1}^i(p)$, *1 ≤ k < np$_i$.*

Property 5.6: *Let p be an nD-OPP. Hence* $p = \bigcup_{k}^{np_i - 1} Slice_k^i(p)$.

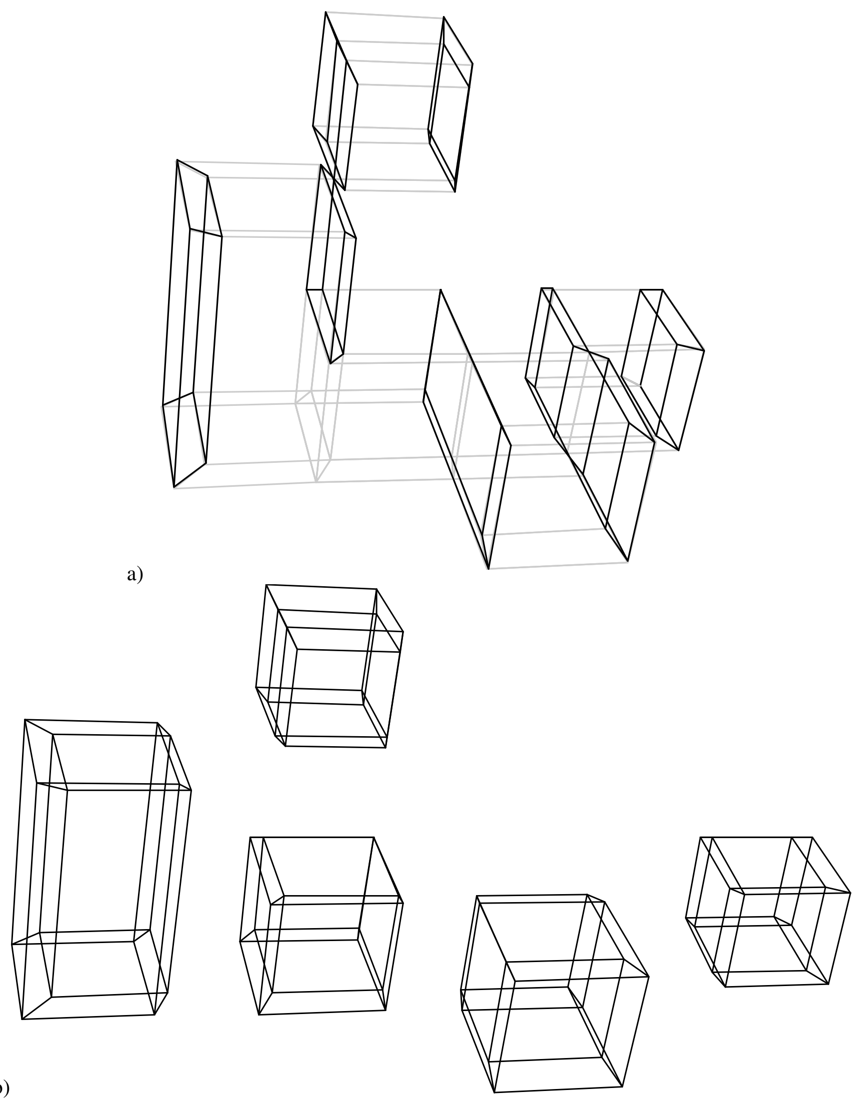
Figure 5.12 shows the slices for a 3D-OPP according to the supporting planes of its 2D-couplets perpendicular to X_1-axis.



a) b)

Figure 5.12. The slices of a 3D-OPP (presented in **Figure 5.5**). b) There are presented the regions from the 3D-OPP between the supporting planes of a) the 2D-couplets perpendicular to X_1-axis

In the **Figure 5.13.a** are shown the regions between the 3D-couplets perpendicular to X_1-axis of the 4D-OPP presented in **Figure 5.6.2**. Finally, in the **Figure 5.13.b** are shown the 4D-OPP's slices.

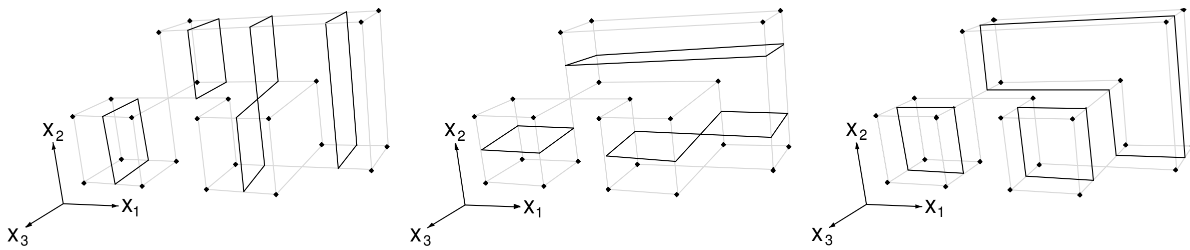


a)

b)

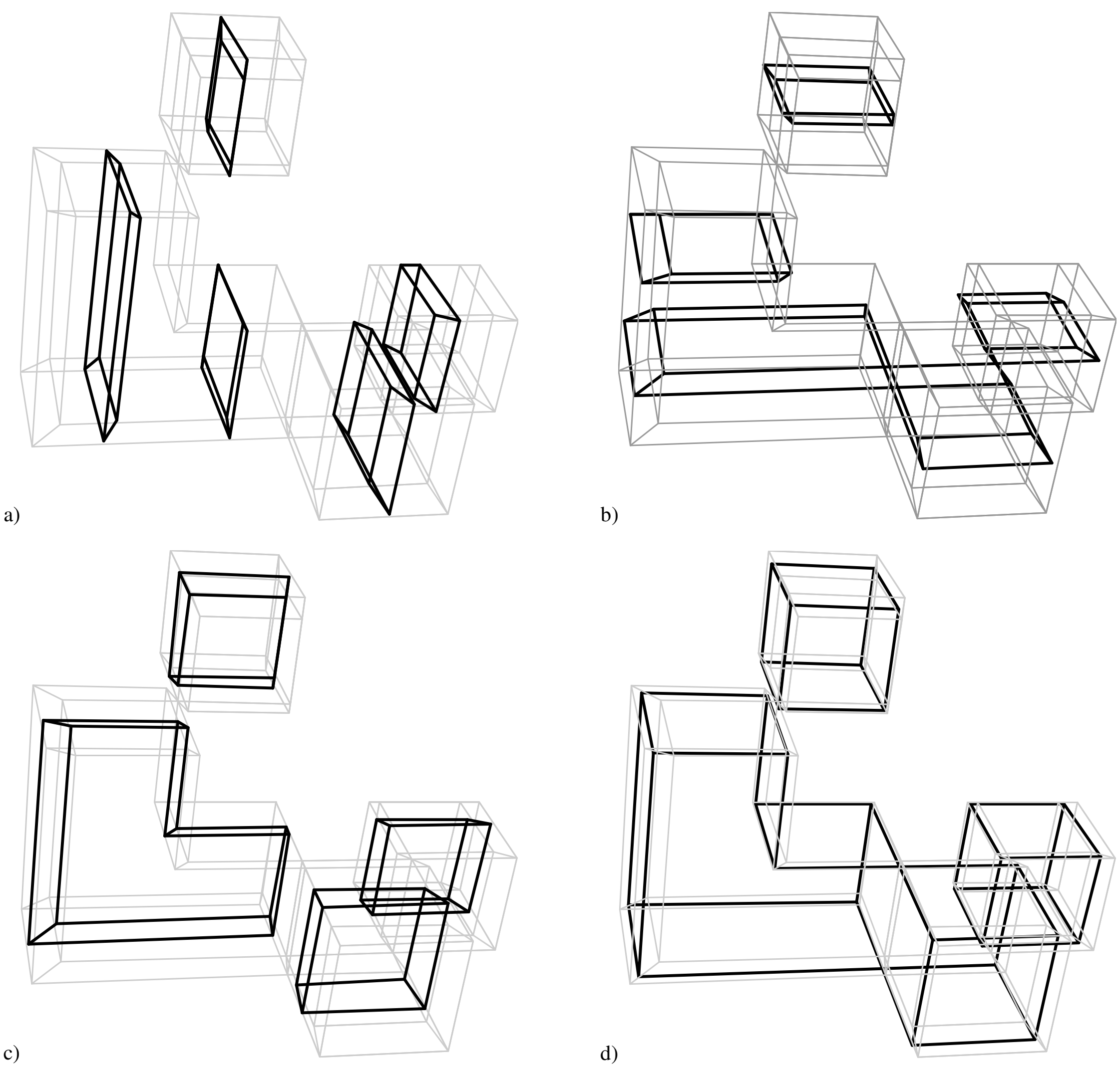
Figure 5.13. The regions of a 4D-OPP (presented in **Figure 5.6.2**) between its couplets perpendicular to X_1-axis (a) and its respective slices (b).

Definition 5.17: *A* <u>*Section*</u> *is the (n-1)D-OPP, $n > 1$, resulting from the intersection between an nD-OPP p and a (n-1)D hyperplane perpendicular to the coordinate axis X_i, $1 \geq i \geq n$, which not coincide with any (n-1)D-couplet of p. A section will be called* <u>*external*</u> *or* <u>*internal*</u> *section of p if it is empty or not, respectively.* $\underline{S_k^i(p)}$ *will refer to the k-th section of p between $\Phi_k^i(p)$ and $\Phi_{k+1}^i(p)$, $1 \leq k < np_i$. Moreover,* $\underline{S_0^i(p)}$ *and* $\underline{S_{np_i}^i(p)}$ *will refer to the empty sections of p before $\Phi_1^i(p)$ and after of $\Phi_{np_i}^i(p)$, respectively. Finally,* $\underline{ns_i}$ *$= np_i + 1$refers to the number of sections of the nD-OPP p.*



a) b) c)

Figure 5.14. The sections of a 3D-OPP (the OPP presented in **Figure 5.5**). a) The internal sections perpendicular to X_1-axis. b) The internal sections perpendicular to X_2-axis. c) The internal sections perpendicular to X_3-axis.



a) b) c) d)

Figure 5.15. A 4D-OPP (from **Figure 5.6.2**) and its sections perpendicular to X_1 (a), X_2 (b), X_3 (c) and X_4 (d) axes.

For example, in **Figure 5.14** are shown the internal sections of a 3D-OPP while in **Figure 5.15** are respectively shown:

- The sections perpendicular to X_1-axis (**5.15.a**);
- The sections perpendicular to X_2-axis (**5.15.b**);
- The sections perpendicular to X_3-axis (**5.15.c**);
- The sections perpendicular to X_4-axis (**5.15.d**);

From the 4D-OPP presented in **Figure 5.6.2**.

Property 5.7: *Let p be an nD-OPP. All the (n-1)D hyperplanes perpendicular to X_i-axis, $1 \le i \le n$, which intersect to $Slice_k^i(p)$ give the same section $S_k^i(p)$.*

5.5. Relating Sections and Couplets

As commented in the introduction of this chapter, the following sections contain the results previously established by Aguilera & Ayala, in [Aguilera97] and [Aguilera98]. As seen in previous sections, the Odd Edge Characterization provides us a unifying classification for edges that compose brinks in the nD-OPP's which is independent of other topological characterizations (See **Appendices D**, **E** and **F**). By this way Extreme Vertices have been defined under a unique framework which allows us to present Aguilera & Ayala's results under the context of the nD-OPP's, and moreover, most of them are direct extensions of the original 3D-OPP's statements.

In **Section 5.1** we commented that the points that compose to an nD-OPP are obtained through the union of the images of the hyper-boxes in its corresponding n-chain. We have defined sets over an nD-OPP, such as the couplets, as sets of cells that belong to p. Starting from this section, we will require to consider these sets of cells in two senses: 1) when we deal with their composing cells, and 2) when we deal with the points in the images of their composing cells.

5.5.1. The Neighborhood of a Couplet

Definition 5.18: *Let $\underline{\varepsilon \in \mathbb{R}^+}$ be a value small enough so that $a_k + \varepsilon < a_{k+1}$, $\forall a_k \in VX_i$, $1 \le i \le n$.*

Definition 5.19: *For each $\underline{q} \in H\left(\Phi_k^i(p)\right)$ we define two points $\underline{q^-}$ and $\underline{q^+}$ having the same coordinates than q and whose x_i coordinates are $a_k - \varepsilon$ and $a_k + \varepsilon$ respectively.*

Lemma 5.7: *Let P be an nD-OPP.* $q^- \in P \Rightarrow q^- \in S_{k-1}^i(P)$ *and* $q^+ \in P \Rightarrow q^+ \in S_k^i(P)$.

Proof [Aguilera98]:

Let Q^- and Q^+ be two parallel hyperplanes to $H\left(\Phi_k^i(P)\right)$ whose equations are $x_i = a_k - \varepsilon$ and $x_i = a_k + \varepsilon$ respectively, thus $q^- \in Q^-$ and $q^+ \in Q^+$. From **Definition 5.16**, $Slice_{k-1}^i(P)$ and $Slice_k^i(P)$ are the regions of P immediately before and after $H\left(\Phi_k^i(P)\right)$ respectively, then, by construction of q^- and q^+, it follows that $q^- \in P \Rightarrow q^- \in Slice_{k-1}^i(P)$ and $q^+ \in P \Rightarrow q^+ \in Slice_k^i(P)$. According to **Property 5.7**, $Slice_{k-1}^i(P) \cap Q^- = S_{k-1}^i(P)$ and $Slice_k^i(P) \cap Q^+ = S_k^i(P)$, therefore $q^- \in P \Rightarrow q^- \in S_{k-1}^i(P)$ and $q^+ \in P \Rightarrow q^+ \in S_k^i(P)$. ◘

Lemma 5.8: *Let P be an nD-OPP. q ∈ P if and only if* $q^- \in S_{k-1}^i(P)$ *or* $q^+ \in S_k^i(P)$.

Proof [Aguilera98]:

⇒)

If q ∈ P then by assuming that both $q^- \notin S_{k-1}^i(P)$ and $q^+ \notin S_k^i(P)$ we get that q is on a dangling kD element of P, $0 \le k < n$, and this lead us to conclude that P is not regular, that is, P is not an nD-OPP. Thus $q \in P \Rightarrow (q^- \in S_{k-1}^i(P)$ or $q^+ \in S_k^i(P))$.

$\Leftarrow$)

Because P is an nD-OPP then P is regular, thus in the limit when ε→0, $q^- \in S^i_{k-1}(P)$ ⇒ q ∈ P and $q^+ \in S^i_k(P)$ ⇒ q ∈ P. Hence ($q^- \in S^i_{k-1}(P)$ or $q^+ \in S^i_k(P)$) ⇒ q ∈ P. ◾

Definition 5.20: *Consider an nD-OPP p.* *<u>T(p)</u> will denote to the set of (n-2)D cells of p such that an (n-2)D cell e_0 belongs to T(p) if and only if e_0 belongs to an (n-1)D cell present in ∂(p), i.e.*

$$e_0 \in T(p) \Leftrightarrow \left(\exists c, c \text{ is present in } \partial(p)\right)\left(e_0\left([0,1]^{n-2}\right) \subseteq c\left([0,1]^{n-1}\right)\right)$$

Property 5.8: *Consider an nD-OPP p.* $\Phi^i_k(p) \cap T(p)$ *is a subset of* $\Phi^i_k(p)$ *with one dimension less that* $\Phi^i_k(p)$.

Lemma 5.9: *Consider an nD-OPP P. The four membership combinations of* q^- *in* $S^i_{k-1}(P)$ *and* q^+ *in* $S^i_k(P)$ *provide the membership characterization of q with respect to* $\Phi^i_k(P)$ *as:*

1. $\left(q^- \in S^i_{k-1}(P) \text{ and } q^+ \in S^i_k(P)\right) \Rightarrow \left(q \notin \Phi^i_k(P)\right) \vee \left(q \in \Phi^i_k(P) \cap T(P)\right)$
2. $\left(q^- \in S^i_{k-1}(P) \text{ and } q^+ \notin S^i_k(P)\right) \Rightarrow q \in \Phi^i_k(P)$
3. $\left(q^- \notin S^i_{k-1}(P) \text{ and } q^+ \in S^i_k(P)\right) \Rightarrow q \in \Phi^i_k(P)$
4. $\left(q^- \notin S^i_{k-1}(P) \text{ and } q^+ \notin S^i_k(P)\right) \Rightarrow q \notin \Phi^i_k(P)$

Proof [Aguilera98]:

The proof is obtained by the exhaustive characterization of all nine possible combinations of the three possible membership cases (in, on or out) of q^- respect to $S^i_{k-1}(P)$ with those of q^+ with respect to $S^i_k(P)$. Cases 1, 2 and 3 will be subdivided (see **Table 5.7**). Case 4 is the counterreciprocal of **Lemma 5.8**, so q ∉ P and thus $q \notin \Phi^i_k(P)$.

Cases 1, 2 and 3 are particular cases of **Lemma 5.8**, therefore q ∈ P. However, case 1.1 with a small value of ε makes point q a interior point of P, so $q \notin \Phi^i_k(P)$. On the other hand, cases 2.1 and 3.1 with small values of ε make their corresponding points q boundary points; therefore q must lie on $\Phi^i_k(P)$, the boundary of P between $S^i_{k-1}(P)$ and $S^i_k(P)$. Cases 1.2 and 1.3 correspond to the limit of cases 2.1 and 3.1 respectively, when q^+ or q^- (the point outside P) approaches to a boundary of P other than $\Phi^i_k(P)$. Cases 2.2 and 3.2 correspond to the limit of cases 2.1 and 3.1 respectively, but when q^- or q^+ (the point inside P) approaches to a boundary of P other than $\Phi^i_k(P)$. Any way, in cases 1.2, 1.3, 2.1, 2.2, 3.1 and 3.2, it is verified that q ∈ $\Phi^i_k(P)$. Case 1.4 corresponds to the limit of case 1.1 (also of case 4) when both q^- and q^+, and thus q approach simultaneously to a boundary of P other than $\Phi^i_k(P)$, therefore $q \notin \Phi^i_k(P)$.

Finally, case 1.5 is when q^- and q^+ lie on different boundary elements of P (although they have the same supporting (n-2)D hyperplane), and thus q is on an (n-2)D cell, i.e., q ∈ T(P), but also inside $\Phi^i_k(P)$. Therefore, in this case $q \in \Phi^i_k(P) \cap T(P)$.

◾

Case	q^- vs. $S_{k-1}^i(P)$	q^+ vs. $S_k^i(P)$	q vs. $\Phi_k^i(P)$
1.1	In	In	Out
1.2	In	On	On
2.1	In	Out	In
1.3	On	In	On
1.4/1.5	On	On	Out/In
2.2	On	Out	On
3.1	Out	In	In
3.2	Out	On	On
4	Out	Out	Out

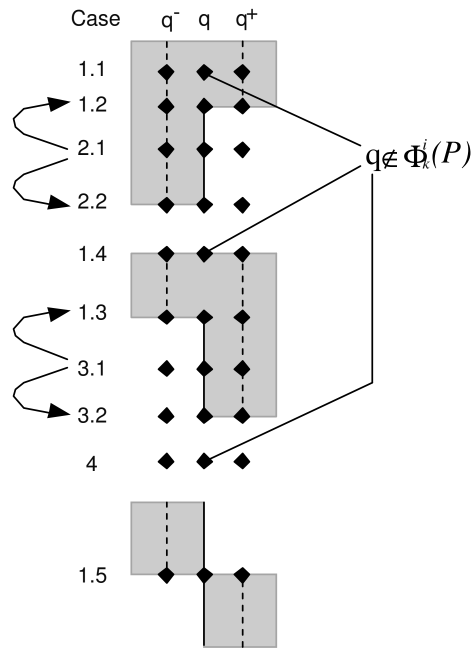


Table 5.7. The membership of q^- and q^+ with respect to $S_{k-1}^i(P)$ and $S_k^i(P)$ respectively, and the resulting membership of q with respect to $\Phi_k^i(P)$ (Solid vertical lines represent $\Phi_k^i(P)$ for some k, while dashed lines on its left and right represent sections $S_{k-1}^i(P)$ and $S_k^i(P)$ respectively).

5.5.2. Computing Couplets from Sections

Theorem 5.16: *The projection of the set of (n-1)D-couplets, $\pi_i\left(\Phi_k^i(P)\right)$, $1 \le i \le n$, of an nD-OPP P, can be obtained by computing the regularized XOR (⊗) between the projections of its previous $\pi_i\left(S_{k-1}^i(P)\right)$ and next $\pi_i\left(S_k^i(P)\right)$ sections, i.e.,*

$$\pi_i\left(\Phi_k^i(P)\right) = \pi_i\left(S_{k-1}^i(P)\right) \otimes^* \pi_i\left(S_k^i(P)\right),\ \forall k \in [1, np_i]$$

Proof [Aguilera98]:
We will prove the double inclusion using **Definition 5.19** and **Lemma 5.9**, and the fact that the projections of q, q^- and q^+ are all the same, i.e., $\pi_i(q) = \pi_i(q^+) = \pi_i(q^-)$.

$\supseteq$)

$q^- \in S_{k-1}^i(P)$ and $q^+ \notin S_k^i(P) \Rightarrow q \in \Phi_k^i(P)$ (case 2 of **Lemma 5.9**)

$\pi_i(q) \in \pi_i\left(S_{k-1}^i(P)\right)$ and $\pi_i(q) \notin \pi_i\left(S_k^i(P)\right) \Rightarrow \pi_i(q) \in \pi_i\left(\Phi_k^i(P)\right)$ (applying the same projection to each element)

$\pi_i(q) \in \pi_i\left(S_{k-1}^i(P)\right) - \pi_i\left(S_k^i(P)\right) \Rightarrow \pi_i(q) \in \pi_i\left(\Phi_k^i(P)\right)$ (by definition of the difference operator)

$\pi_i\left(S_{k-1}^i(P)\right) -^* \pi_i\left(S_k^i(P)\right) \subseteq \pi_i\left(\Phi_k^i(P)\right)$ (a) (since it is valid $\forall \pi_i(q) \in \pi_i\left(S_{k-1}^i(P)\right) - \pi_i\left(S_k^i(P)\right)$)

Similarly, case 3 of **Lemma 5.9** implies that $\pi_i\left(S_k^i(P)\right) -^* \pi_i\left(S_{k-1}^i(P)\right) \subseteq \pi_i\left(\Phi_k^i(P)\right)$ (b)

Thus, by (a) and (b), $\pi_i\left(S_{k-1}^i(P)\right) -^* \pi_i\left(S_k^i(P)\right) \cup^* \pi_i\left(S_k^i(P)\right) -^* \pi_i\left(S_{k-1}^i(P)\right) \subseteq \pi_i\left(\Phi_k^i(P)\right)$.

Therefore $\pi_i\left(S_{k-1}^i(P)\right) \otimes^* \pi_i\left(S_k^i(P)\right) \subseteq \pi_i\left(\Phi_k^i(P)\right),\ \forall k \in [1, np_i]$

$\subseteq$)

The contrareciprocal of case 4 of **Lemma 5.9** is $q \in \Phi_k^i(P) \Rightarrow q^- \in S_{k-1}^i(P)\ or\ q^+ \in S_k^i(P)$

and it implies that $\pi_i\left(\Phi_k^i(P)\right) \subseteq \pi_i\left(S_{k-1}^i(P)\right) \cup^* \pi_i\left(S_k^i(P)\right)$ (c)

with a procedure similar to the above. Similarly negation of case 1 of **Lemma 5.9** is:

$(q \in \Phi_k^i(P) \ and \ q \notin \Phi_k^i(P) \cap T(P)) \Rightarrow (q^- \notin S_{k-1}^i(P) \ or \, q^+ \notin S_k^i(P))$, i.e.,

$\pi_i(q) \in \pi_i\left(\Phi_k^i(P)\right) - \pi_i\left(\Phi_k^i(P) \cap T(P)\right) \Rightarrow \pi_i(q) \notin \pi_i\left(S_{k-1}^i(P)\right) \ or \, \pi_i(q) \notin \pi_i\left(S_k^i(P)\right)$ (applying projections)

$\Rightarrow \pi_i(q) \in \left(\pi_i\left(S_{k-1}^i(P)\right)\right)^C \ or \, \pi_i(q) \in \left(\pi_i\left(S_k^i(P)\right)\right)^C$ (the complement sets)

$\Rightarrow \pi_i(q) \in \left(\pi_i\left(S_{k-1}^i(P)\right)\right)^C \cup \left(\pi_i\left(S_k^i(P)\right)\right)^C$ (by definition of union operator)

Thus

$\pi_i(q) \in \pi_i\left(\Phi_k^i(P)\right) - \pi_i\left(\Phi_k^i(P) \cap T(P)\right) \Rightarrow \pi_i(q) \in \left(\pi_i\left(S_{k-1}^i(P) \cap S_k(P)\right)\right)^C$ (by DeMorgan's law)

This is true $\forall \pi_i(q) \in \pi_i\left(\Phi_k^i(P)\right) - \pi_i\left(\Phi_k^i(P) \cap T(P)\right)$

Hence $\pi_i\left(\Phi_k^i(P)\right) - \pi_i\left(\Phi_k^i(P) \cap T(P)\right) \subseteq \left(\pi_i\left(S_{k-1}^i(P) \cap S_k(P)\right)\right)^C$

Now, by regularizing both sides, and observing that $\pi_i\left(\Phi_k^i(P)\right) -^* \pi_i\left(\Phi_k^i(P) \cap T(P)\right) = \pi_i\left(\Phi_k^i(P)\right)$ since $\Phi_k^i(P) \cap T(P$ is a subset of $\Phi_k^i(P)$ with one dimension less that $\Phi_k^i(P)$, see **Property 5.8**, then

$\pi_i\left(\Phi_k^i(P)\right) \subseteq \left(\pi_i\left(S_{k-1}^i(P) \ \cap^* \ S_k^i(P)\right)\right)^C$ (d) Hence

$\pi_i\left(\Phi_k^i(P)\right) \subseteq \pi_i\left(S_{k-1}^i(P) \ \cup^* \ S_k^i(P)\right) \ \cap^* \ \left(\pi_i\left(S_{k-1}^i(P) \ \cap^* \ S_k^i(P)\right)\right)^C$ (by (c) and (d))

$\pi_i\left(\Phi_k^i(P)\right) \subseteq \pi_i\left(S_{k-1}^i(P) \ \cup^* \ S_k^i(P)\right) \ -^* \ \pi_i\left(S_{k-1}^i(P) \ \cap^* \ S_k^i(P)\right)$ (by definition of difference operator)

$\pi_i\left(\Phi_k^i(P)\right) \subseteq \pi_i\left(S_{k-1}^i(P)\right) \ \otimes^* \ \pi_i\left(S_k^i(P)\right), \ \forall k \in [1, np_i]$ (by definition of XOR operator)

$\therefore \ \pi_i\left(\Phi_k^i(P)\right) = \pi_i\left(S_{k-1}^i(P)\right) \ \otimes^* \ \pi_i\left(S_k(P)\right), \ \forall k \in [1, np_i]$ ◪

5.5.3. Computing Sections from Couplets

Theorem 5.17: *The projection of any section,* $\pi_i\left(S_k^i(p)\right)$, *of an nD-OPP p, can be obtained by computing the regularized XOR between the projection of its previous section,* $\pi_i\left(S_{k-1}^i(p)\right)$, *and the projection of its previous couplet* $\pi_i\left(\Phi_k^i(p)\right)$. *Or, equivalently, by computing the regularized XOR of the projections of all the previous couplets, i.e.*

$$\begin{cases} S_0^i(p) = \varnothing \\ \pi_i\left(S_k^i(p)\right) = \pi_i\left(S_{k-1}^i(p)\right) \ \otimes^* \ \pi_i\left(\Phi_k^i(p)\right), \ \forall k \in [1, np_i] \end{cases} \quad \text{that is} \quad \pi_i\left(S_k^i(p)\right) = \bigotimes_{j=1}^{k}{}^{*} \, \pi_i\left(\Phi_j^i(p)\right)$$

Proof [Aguilera98]:

By **Definition 5.17**, $S_0^i(p) = \varnothing$. The resulting equation of **Theorem 5.16** can be solved for $\pi_i\left(S_k^i(p)\right)$, giving the recursive equation of this theorem. ◪

Corollary 5.5: *The projection of the first and last couplets of any nD-OPP p, must coincide with the projection of the first and last internal section of p, that is*

$$\pi_i\left(S_1^i(p)\right) = \pi_i\left(\Phi_1^i(p)\right) \ and \ \pi_i\left(S_{np_i-1}^i(p)\right) = \pi_i\left(\Phi_{np_i}^i(p)\right)$$

Proof [Aguilera98]:

Since $S_0^i(p) = \varnothing$ then by **Theorem 5.17**, $\pi_i\left(S_1^i(p)\right) = \pi_i\left(\Phi_1^i(p)\right)$. Similarly, since $S_{np_i}^i(p) = \varnothing$ then $S_{np_i}^i(p) = \pi_i\left(S_{np_i-1}^i(p)\right) \ \otimes^* \ \pi_i\left(\Phi_{np_i}^i(p)\right) = \varnothing$ if and only if $\pi_i\left(S_{np_i-1}^i(p)\right) = \pi_i\left(\Phi_{np_i}^i(p)\right)$ ◪

Lets to consider an example where we will apply the **Theorems 5.16** and **5.17**. Consider a 3D-OPP which is the union of four 3D-OPP's as shown in **Figure 5.16.a**. The faces parallel to X_3-axis are the same in each one of these 3D-OPP's (hence, these OPP's can be seen as extrusions towards 3D space of 2D-OPP's). **Figure 5.16.b** shows the final 3D-OPP.

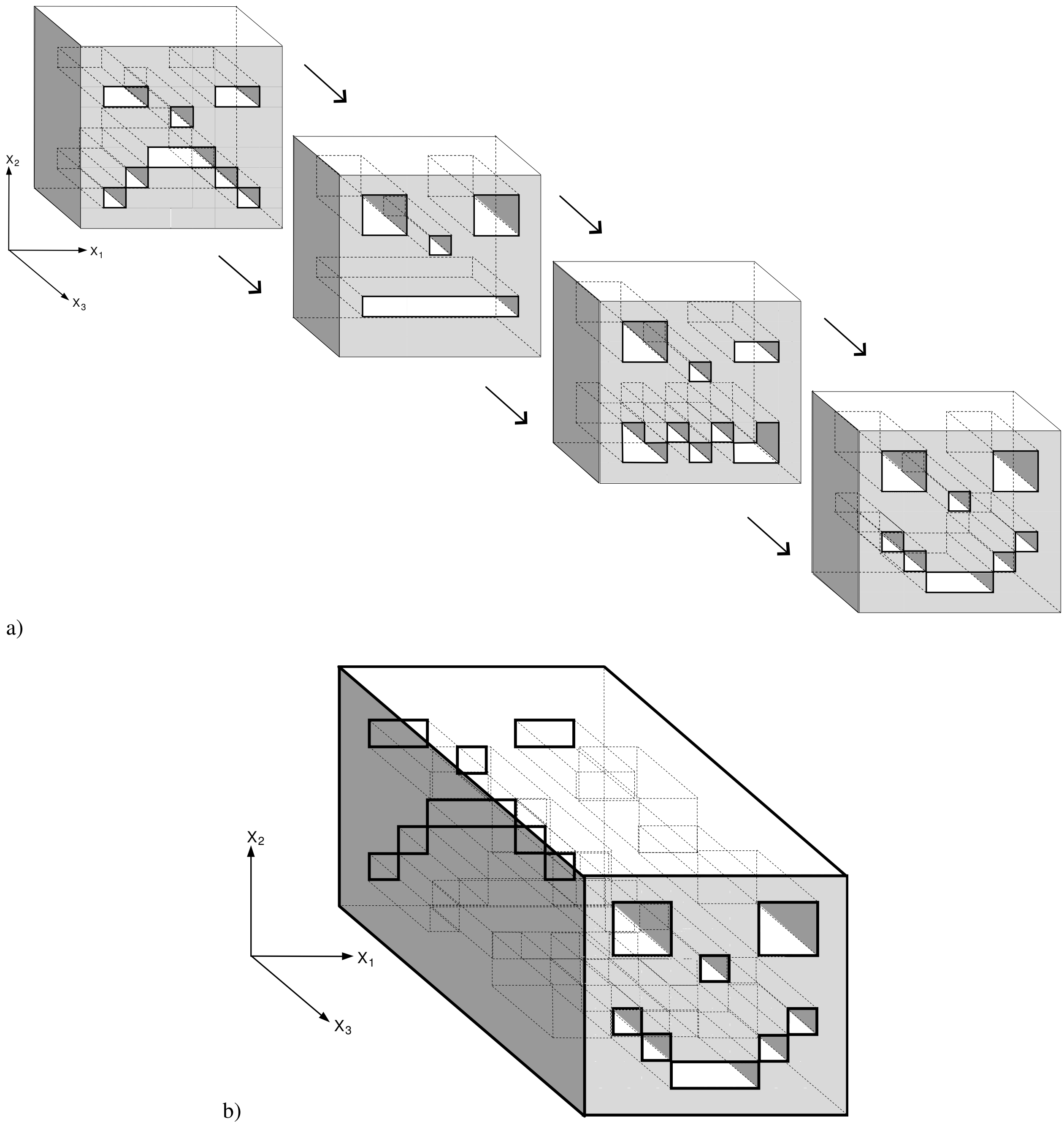


Figure 5.16. a) Composing a 3D-OPP as the union of four 3D-OPP's. b) The Final 3D-OPP.

We can identify, through **Figure 5.16.a**, that there are five distinct coordinates in X_3-axis, hence, we have to expect five 2D-couplets, i.e., Extended Faces, which are shown in **Figure 5.17**. Such couplets are embedded in planes (referenced in the figure through dotted squares) which are also perpendicular to X_3-axis. **Figure 5.18** shows the application of the projection operator over these couplets. The projection takes place in a 2D space whose normal vector is perpendicular to X_3-axis.

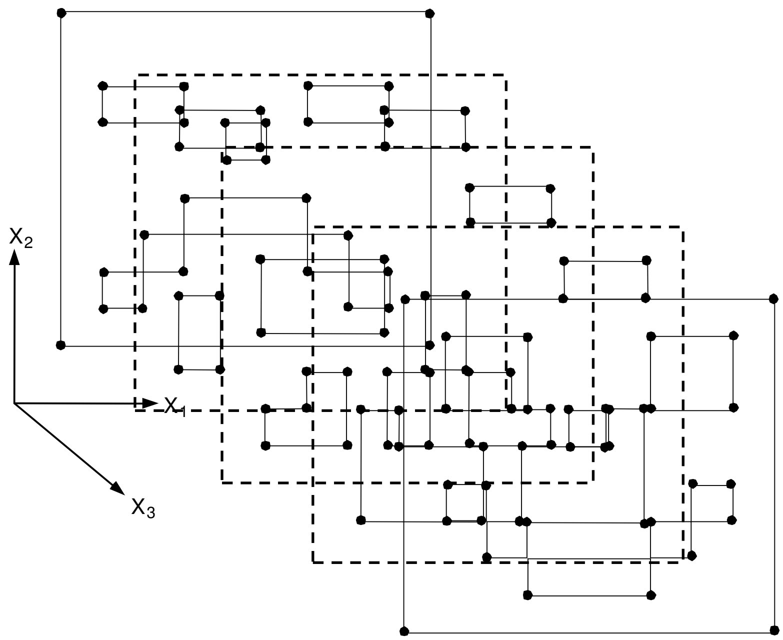


Figure 5.17. Disposition in the 3D space of couplets perpendicular to X_3-axis of the 3D-OPP shown in **Figure 5.16.a**.

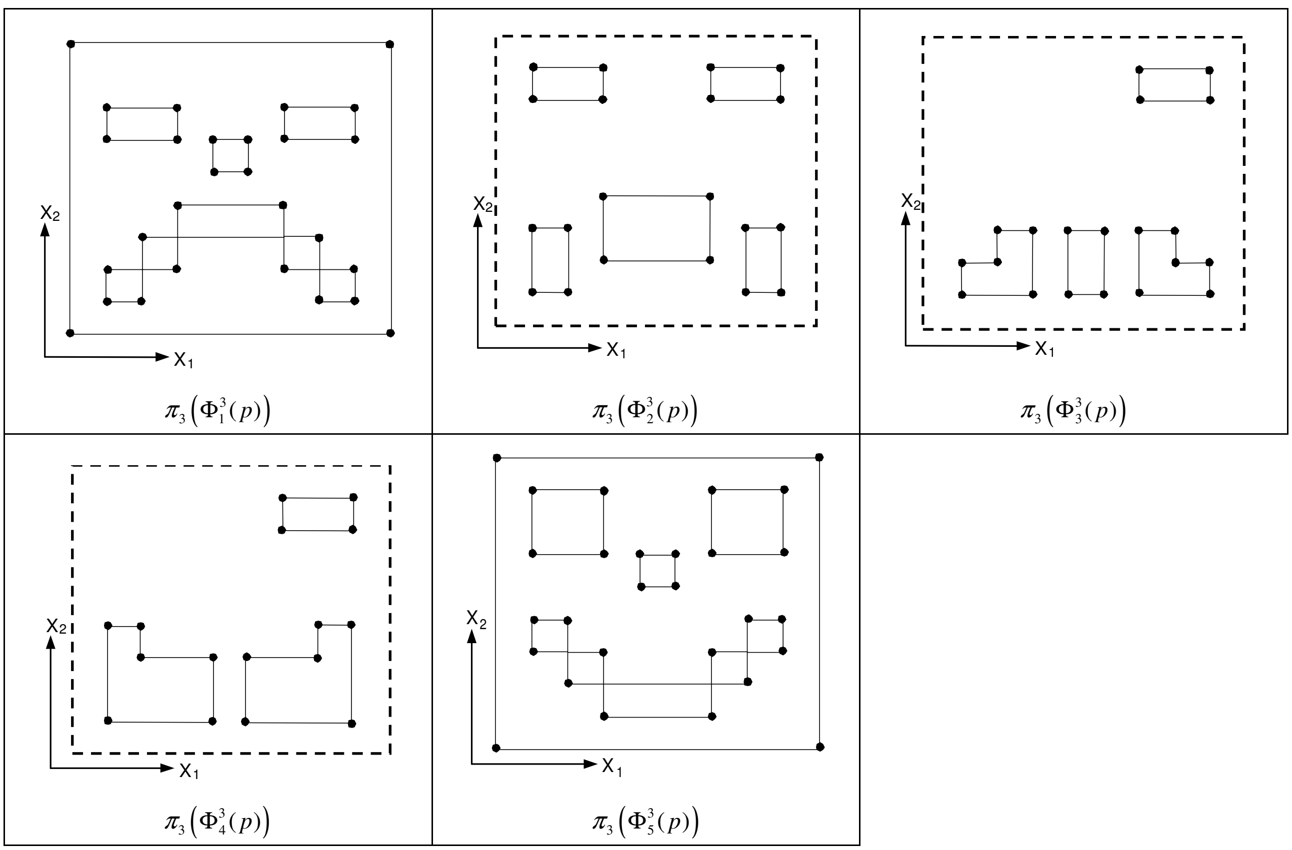


Figure 5.18. The Projections of couplets of the 3D-OPP shown in **Figure 5.16.a**.

Table 5.8 shows the application of **Theorem 5.17** in order to get the set of the projections of sections perpendicular to X_3-axis. Once we have obtained sections from couplets, we can apply **Theorem 5.16** in order to obtain the 3D-OPP's couplets from the sections. The obtained results in **Table 5.9** coincide with **Figure 5.18**.

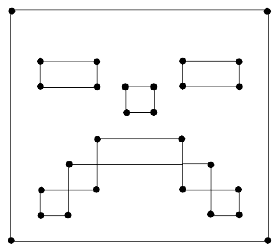
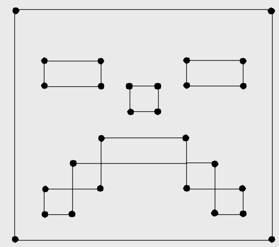
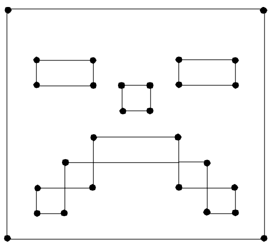
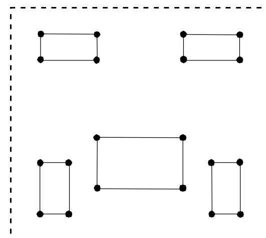
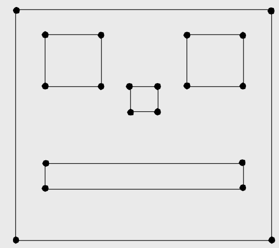
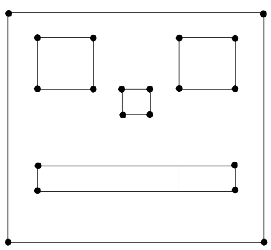
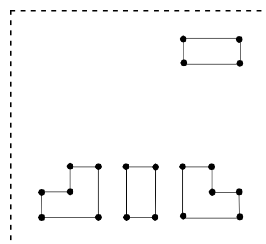
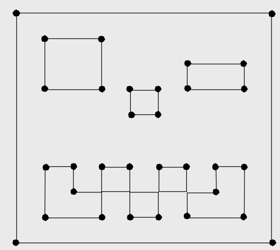
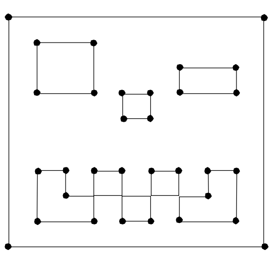
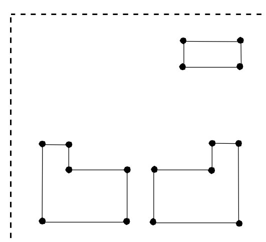
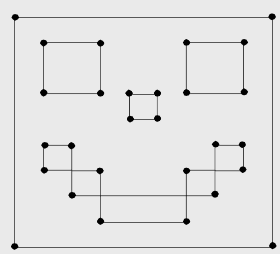
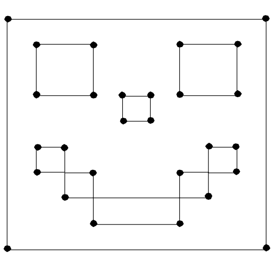
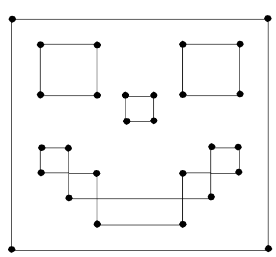
$\pi_3\left(S_{k-1}^3(p)\right)$	$\pi_3\left(\Phi_k^3(p)\right)$	$\pi\left(S_k^3(p)\right) =$ $\pi\left(S_{k-1}^3(p)\right) \otimes^* \pi\left(\Phi_k^3(p)\right)$
$S_0^3(p) = \varnothing$	 $\pi_3\left(\Phi_1^3(p)\right)$	 $\pi\left(S_1^3(p)\right) = \pi\left(S_0^3(p)\right) \otimes^* \pi\left(\Phi_1^3(p)\right)$
 $\pi\left(S_1^3(p)\right)$	 $\pi_3\left(\Phi_2^3(p)\right)$	 $\pi\left(S_2^3(p)\right) = \pi\left(S_1^3(p)\right) \otimes^* \pi\left(\Phi_2^3(p)\right)$
 $\pi\left(S_2^3(p)\right)$	 $\pi_3\left(\Phi_3^3(p)\right)$	 $\pi\left(S_3^3(p)\right) = \pi\left(S_2^3(p)\right) \otimes^* \pi\left(\Phi_3^3(p)\right)$
 $\pi\left(S_3^3(p)\right)$	 $\pi_3\left(\Phi_4^3(p)\right)$	 $\pi\left(S_4^3(p)\right) = \pi\left(S_3^3(p)\right) \otimes^* \pi\left(\Phi_4^3(p)\right)$
 $\pi\left(S_4^3(p)\right)$	 $\pi_3\left(\Phi_5^3(p)\right)$	$\pi\left(S_5^3(p)\right) =$ $\pi\left(S_4^3(p)\right) \otimes^* \pi\left(\Phi_5^3(p)\right) = \varnothing$

Table 5.8. Computing sections from couplets of the 3D-OPP shown in **Figure 5.16.a.**

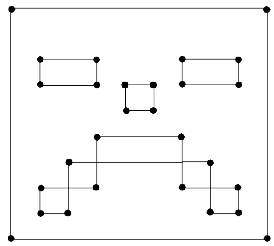
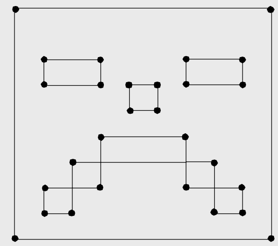
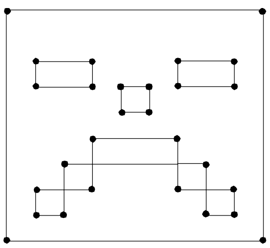
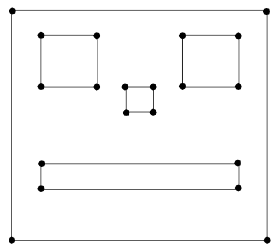
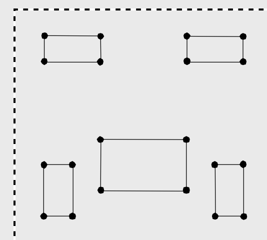
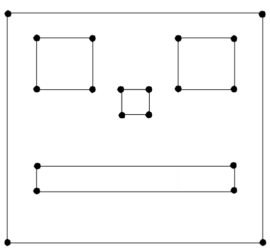
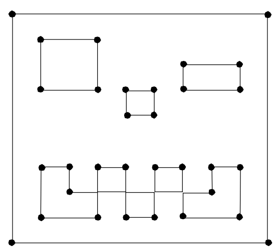
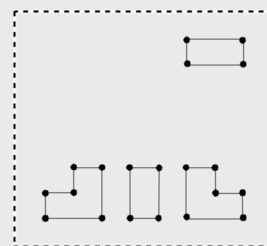
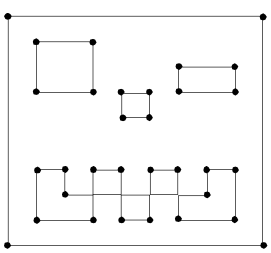
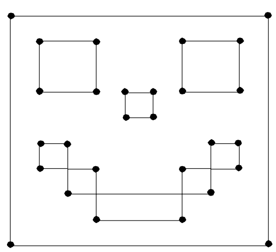
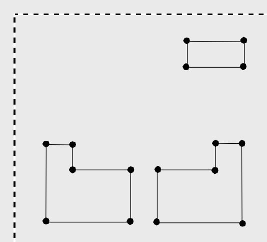
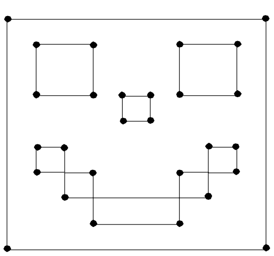
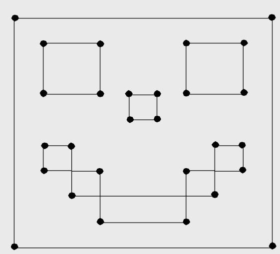
$\pi_3\left(S^3_{k-1}(p)\right)$	$\pi_3\left(S^3_{k}(p)\right)$	$\pi_3\left(\Phi^3_k(p)\right) =$ $\pi_3\left(S^3_{k-1}(p)\right) \otimes^* \pi_3\left(S^3_{k}(p)\right)$
$S^3_0(p) = \varnothing$	 $\pi_3\left(S^3_1(p)\right)$	 $\pi_3\left(\Phi^3_1(p)\right) = S^3_0(p) \otimes^* \pi_3\left(S^3_1(p)\right)$
 $\pi_3\left(S^3_1(p)\right)$	 $\pi_3\left(S^3_2(p)\right)$	 $\pi_3\left(\Phi^3_2(p)\right) = \pi_3\left(S^3_1(p)\right) \otimes^* \pi_3\left(S^3_2(p)\right)$
 $\pi_3\left(S^3_2(p)\right)$	 $\pi_3\left(S^3_3(p)\right)$	 $\pi_3\left(\Phi^3_3(p)\right) = \pi_3\left(S^3_2(p)\right) \otimes^* \pi_3\left(S^3_3(p)\right)$
 $\pi_3\left(S^3_3(p)\right)$	 $\pi_3\left(S^3_4(p)\right)$	 $\pi_3\left(\Phi^3_4(p)\right) = \pi_3\left(S^3_3(p)\right) \otimes^* \pi_3\left(S^3_4(p)\right)$
 $\pi_3\left(S^3_4(p)\right)$	$S^3_5(p) = \varnothing$	 $\pi_3\left(\Phi^3_5(p)\right) = \pi_3\left(S^3_4(p)\right) \otimes^* \pi_3\left(S^3_5(p)\right)$

Table 5.9. Computing couplets from sections of the 3D-OPP shown in **Figure 5.16.a.**

5.5.4. Forward and Backward Differences

According to **Theorem 5.16**, the projection of any couplet, $\pi_i\left(\Phi_k^i(p)\right)$, of an nD-OPP p, can be obtained by computing a regularized XOR between the projections of its previous $\pi_i\left(S_{k-1}^i(p)\right)$ and next $\pi_i\left(S_k^i(p)\right)$ sections, that is

$$\pi_i(\Phi_k^i(p)) = \pi_i(S_{k-1}^i(p)) \otimes^* \pi_i(S_k^i(p)),\ \forall k \in [1, np_i]$$

Where $\Phi_k^i(p)$ is an (n-1)D-OPP representing the set of coupled (n-1)D cells, included in ∂(p), which are embedded in the hyperplane $H\left(\Phi_k^i(p)\right)$. Moreover, according to the well known definition of the XOR operator, we have the following

Property 5.9: *Let p be an nD-OPP. Hence*

$$\pi_i\left(\Phi_k^i(p)\right) = \pi_i\left(S_{k-1}^i(p)\right) \otimes^* \pi_i\left(S_k^i(p)\right) = \left(\pi_i\left(S_{k-1}^i(p)\right) -^* \pi_i\left(S_k^i(p)\right)\right) \cup^* \left(\pi_i\left(S_k^i(p)\right) -^* \pi_i\left(S_{k-1}^i(p)\right)\right)$$

Definition 5.21: *Consider an nD-OPP p.* $\left(\pi_i\left(S_{k-1}^i(p)\right) -^* \pi_i\left(S_k^i(p)\right)\right)$ *and* $\left(\pi_i\left(S_k^i(p)\right) -^* \pi_i\left(S_{k-1}^i(p)\right)\right)$ *will be called* <u>*Forward and Backward Differences*</u> *of the projection of two consecutive sections* $\pi_i\left(S_{k-1}^i(p)\right)$ *and* $\pi_i\left(S_k^i(p)\right)$ *and they will be denoted by* $\underline{FD_k^i(p)}$ *and* $\underline{BD_k^i(p)}$ *where* X_i*-axis is perpendicular to them.*

Property 5.10 [Aguilera97]: *Forward and Backward differences are quasi-disjoint sets to each other, i.e.,* $FD_k^i(p) \cap^* BD_k^i(p) = \varnothing$. *Hence, they define a partition for the set of (n-1)D cells lying on the given couplet* $\Phi_k^i(p)$.

One interesting characteristic, described in [Aguilera98], of forward and backward differences of a 3D-OPP, is that forward differences are the sets of faces, on an couplet, whose normal vectors point to the positive side of the coordinate axis perpendicular to such couplet. Similarly, backward differences are the sets of faces, on a couplet, whose normal vectors point to the negative side of the coordinate axis perpendicular to such couplet. By this way, it is provided a procedure for obtaining the correct orientation of faces in a 3D-OPP when it is converted from 3D-EVM to a boundary representation.

In the context of an nD-OPP's p there are methods to identify normal vectors in the (n-1)D cells included in ∂(p). For example, simplicial combinatorial topology provides methodologies assuming a polytope is represented under a simplexation (see **Section 2.2.4**). Such methods operate under the fact the n+1 vertices of an nD simplex are labeled and sorted. Such sorting corresponds to an odd or an even permutation. By taking n vertices from the n+1 vertices of the nD simplex we get the vertices corresponding to one of its (n-1)D-cells. In this point, usually a set of entirely arbitrary rules are given to determine the normal vector to such (n-1)D cells, see for example [Hocking88] & [Naber00]. Such rules establish, according to the parity of the permutation, if the assigned normal vector points towards the interior of the polytope or outside of it.

On the other hand, there are works that consider the determination of normal vectors by taking in account properties of the vectors that compose the basis of nD space. Let's describe the procedure presented in [Kolcun04]. Consider the 3D simplex s_3, i.e. a tetrahedron, embedded in 3D space, defined by vertices $a_0 = (0,0,0)$, $a_1 = (1,0,0)$, $a_2 = (0,1,0)$ and $a_3 = (0,0,1)$. Let $\vec{e_1}$, $\vec{e_2}$ and $\vec{e_3}$ be the vectors in the well known canonical base for $\mathbb{R}^3$. By applying the definition of cross product we have the following relationships:

$$\vec{e_1} \times \vec{e_2} = \vec{e_3} \qquad \vec{e_3} \times \vec{e_1} = \vec{e_2} \qquad \vec{e_2} \times \vec{e_3} = \vec{e_1}$$

Such identities define the <u>right-orientation of the canonical basis</u> of $\mathbb{R}^3$ [Kolcun04]. The orientation of the cross products of these vectors is expressed in **Figures 5.19.a, b** and **c** as an oriented edge on the boundary of the tetrahedron s_3. So we obtain the <u>right-oriented faces</u> $a_0a_1a_2$, $a_0a_2a_3$ and $a_0a_3a_1$. Because $\vec{e_i} \times \vec{e_j} = -\vec{e_j} \times \vec{e_i}$ then we have the <u>left-oriented faces</u> [Kolcun04] shown in **Figures 5.19.d**, **e** and **f**. Under this orientation, it is well known that faces' normal vectors point outside tetrahedron s_3.

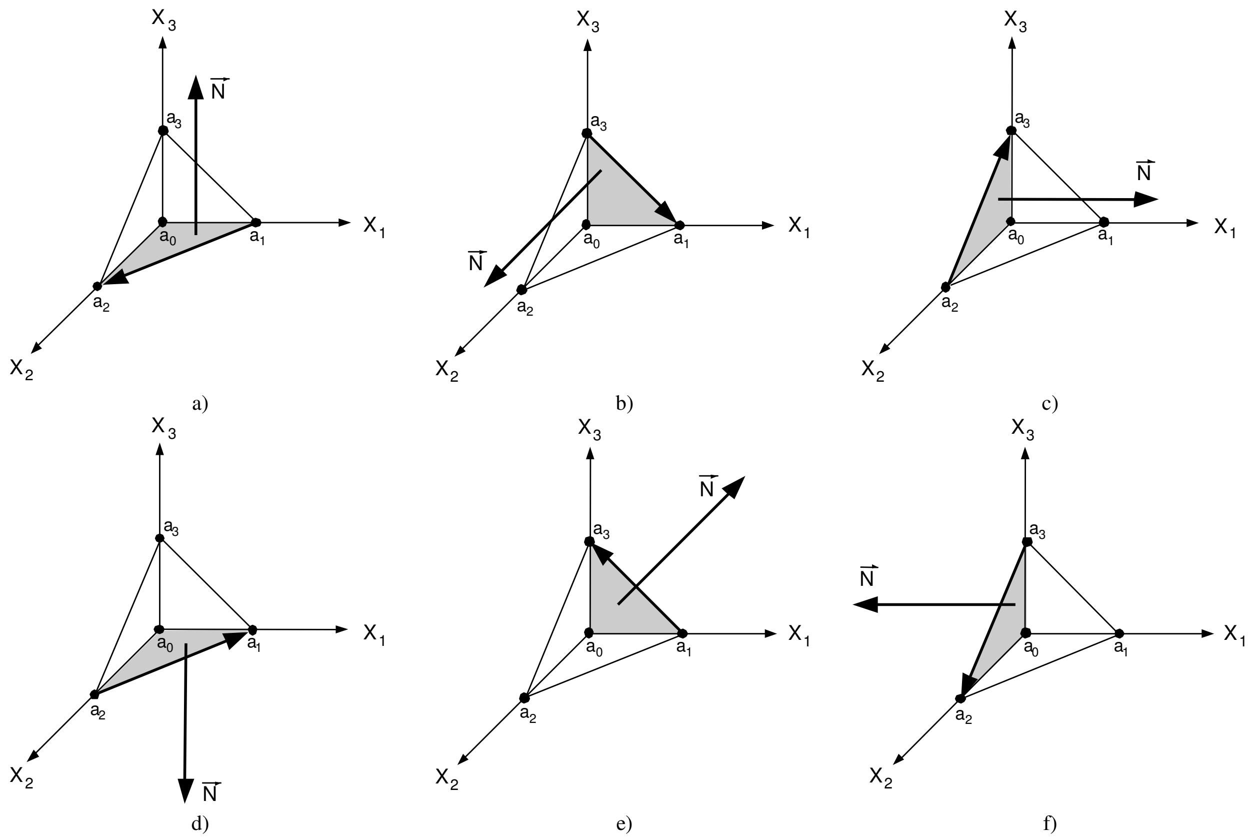


Figure 5.19. a), b) and c) Right-oriented faces $a_0a_1a_2$, $a_0a_2a_3$ and $a_0a_3a_1$ on the boundary of the tetrahedron. In this orientation, normal vector $\vec{N}$ point towards the tetrahedron's interior. d), e) and f) Left-oriented faces on the boundary of the tetrahedron. In this case, normal vector $\vec{N}$ point outside the tetrahedron.

[Kolcun04] extends the previous idea in order to define orientations for volumes on the boundary of a 4D simplex s_4 embedded in 4D space and defined by vertices $a_0 = (0,0,0,0)$, $a_1 = (1,0,0,0)$, $a_2 = (0,1,0,0)$, $a_3 = (0,0,1,0)$ and $a_4 = (0,0,0,1)$. Consider the definition of cross product in four-dimensional space ([Banchoff92] & [Hollasch91]):

$$\vec{u}\times\vec{v}\times\vec{w} = \begin{bmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} & \vec{e_4} \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{bmatrix}$$

Where $\vec{e_1}$, $\vec{e_2}$, $\vec{e_3}$ and $\vec{e_4}$ are the vectors in the canonical base for $\mathbb{R}^4$. Analogously to 3D case the following relationships are fulfilled [Kolcun04]:

$$\vec{e_1} \times \vec{e_3} \times \vec{e_2} = \vec{e_4} \qquad \vec{e_1} \times \vec{e_2} \times \vec{e_4} = \vec{e_3}$$
$$\vec{e_1} \times \vec{e_4} \times \vec{e_3} = \vec{e_2} \qquad \vec{e_2} \times \vec{e_3} \times \vec{e_4} = \vec{e_1}$$

Such relations define the <u>right-orientation of the canonical basis</u> of $\mathbb{R}^4$ [Kolcun04]. The orientation of the cross products of these vectors is expressed in **Figures 5.20.a**, **b**, **c** and **d** as an oriented face. In this way we obtain the <u>right-oriented volumes</u> $a_0a_1a_3a_2$, $a_0a_1a_2a_4$, $a_0a_1a_4a_3$ and $a_0a_2a_3a_4$. In this orientation, volumes' normal vector point towards the 4D simplex's interior. Because $\vec{e_i} \times \vec{e_j} \times \vec{e_k} = -\vec{e_k} \times \vec{e_j} \times \vec{e_i}$ [Hollasch91] then we have the <u>left-oriented volumes</u> [Kolcun04] shown in **Figure 5.20.e**, **f**, **g** and **h**. Under this orientation, we have that volumes' normal vectors point outside simplex s_4.

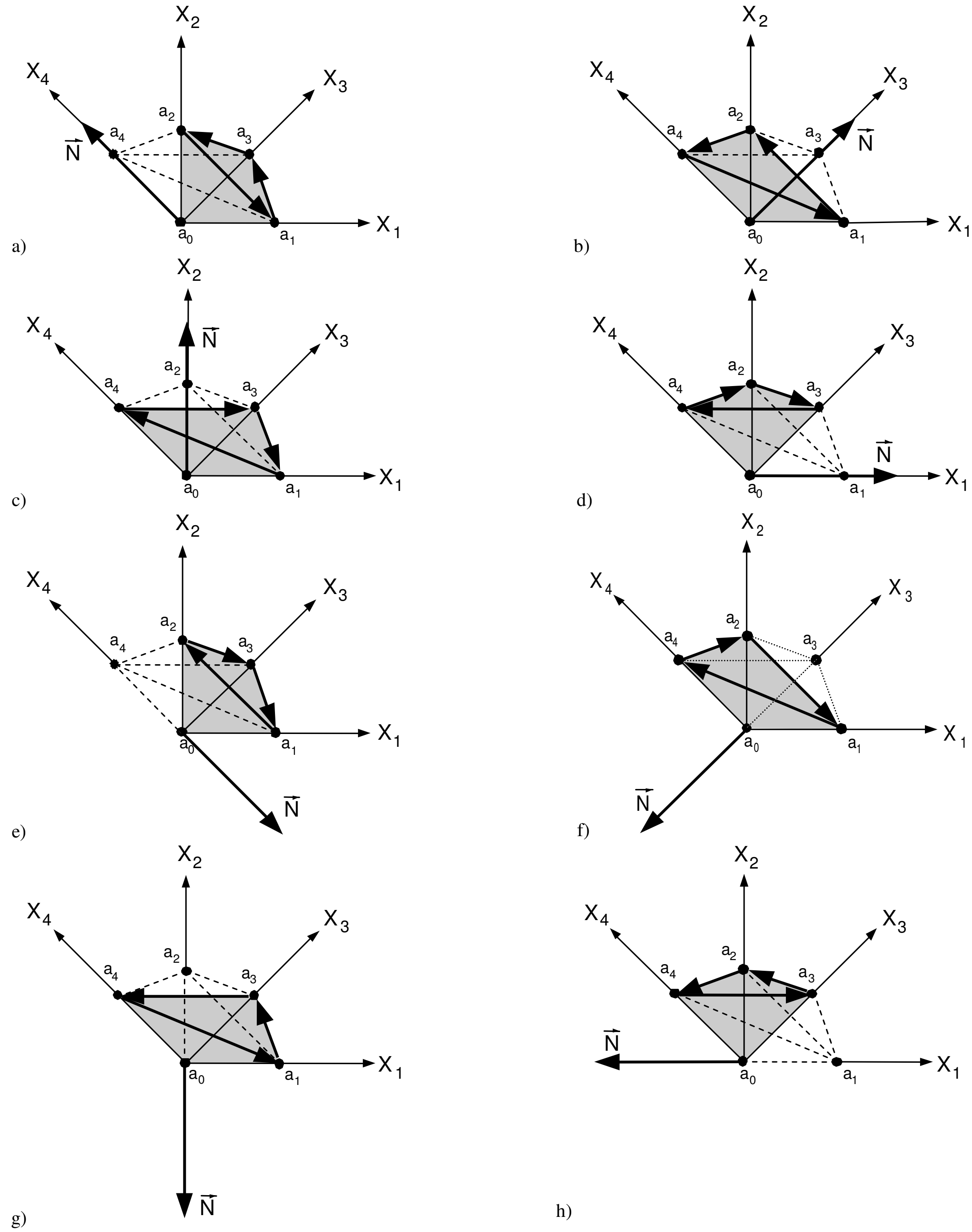


Figure 5.20. a), b), c) and d) Right-oriented volumes on the boundary of the 4D simplex. In this orientation, normal vector $\vec{N}$ point towards simplex's interior. e), f), g) and h) Left-oriented volumes on the boundary of the simplex. In this case, normal vector $\vec{N}$ point outside the simplex.

The above methods, allows us to interpret the direction of the vector product as a direction of the internal normal vector of right-oriented boundary face or volume [Kolcun04]. In our context we will require that normal vectors point outside the polytope, hence, by identities $\vec{e_i} \times \vec{e_j} = -\vec{e_j} \times \vec{e_i}$, in $\mathbb{R}^3$, and $\vec{e_i} \times \vec{e_j} \times \vec{e_k} = -\vec{e_k} \times \vec{e_j} \times \vec{e_i}$, in $\mathbb{R}^4$, we have access to normal vectors pointing outside 3D and 4D simplexes respectively. [Kolcun04] applies this simplicial procedure in order to define normal vectors in the 3D cells of a 4D hypercube. Although Kolcun's method can be extended for nD-OPP's, under the appropriate definitions of cross product in nD space, it is dependent of polytopes are represented through a simplexation. In this last sense, forward and backward differences will provide us a powerful tool.

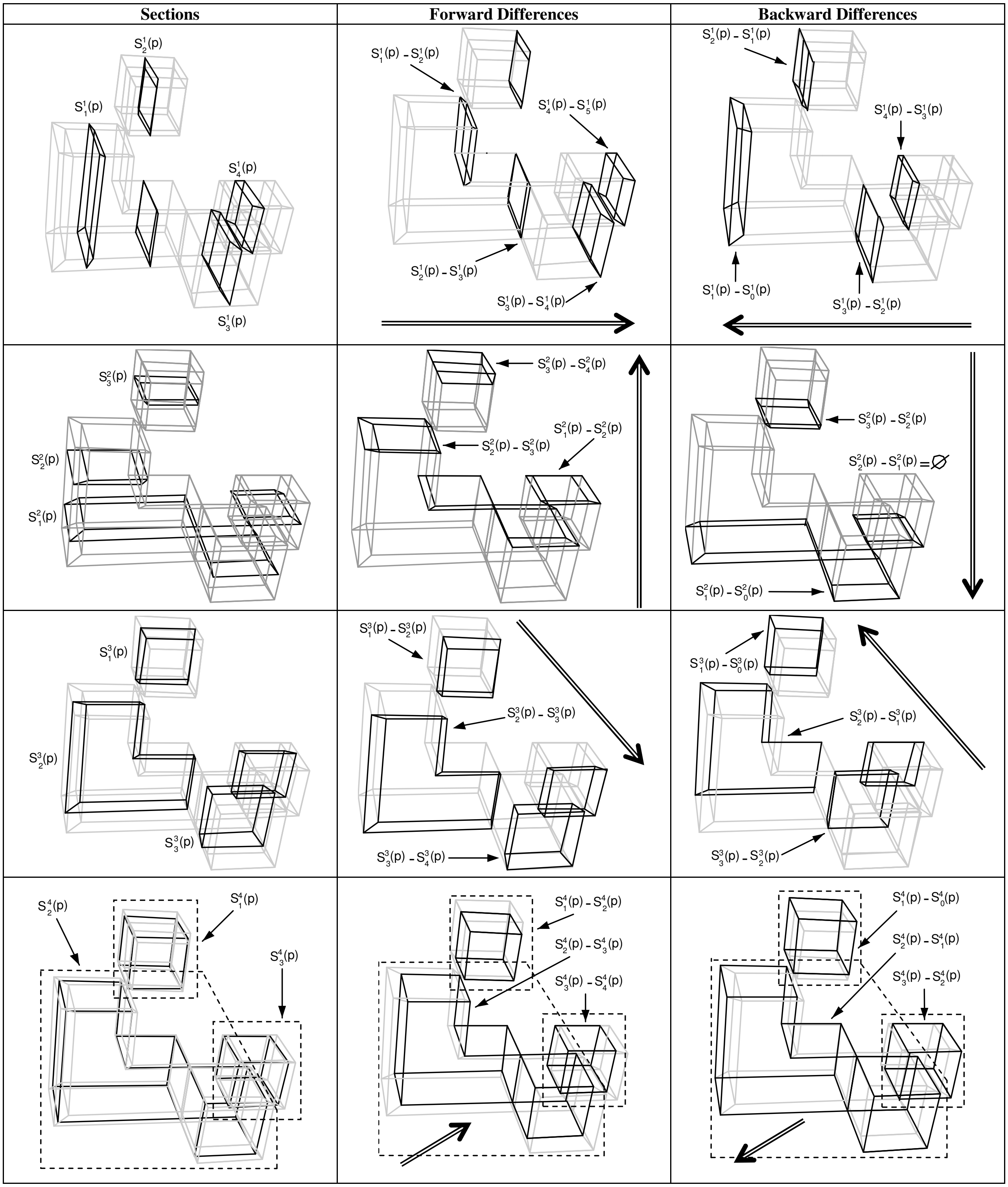


Table 5.10. Computing the Backward and Forward Differences for the 4D-OPP shown in **Figure 5.6.2** (see text for details).

Theorem 5.18: *In an nD-OPP P, forward differences* $FD^i_k(P)$ *are the sets of (n-1)D cells on* $\Phi^i_k(P)$ *whose normal vectors point to the positive side of the coordinate axis* X_i *which is perpendicular to* $\Phi^i_k(P)$*, while backward differences* $BD^i_k(P)$ *are the sets of (n-1)D cells on* $\Phi^i_k(P)$ *whose normal vectors point to the negative side of the coordinate axis* X_i *which is perpendicular to* $\Phi^i_k(P)$.

Proof [Aguilera98]:
From **Definition 5.21**, forward differences $\left(\pi_i\left(S_{k-1}^i(P)\right) -* \ \pi_i\left(S_k^i(P)\right)\right)$ correspond to the set of points q, such that its corresponding points q^- and q^+ satisfy $q^- \in S_{k-1}^i(P)$ and $q^+ \notin S_k^i(P)$, i.e., q^- is inside, and q^+ is outside P. Moreover, q, q^- and q^+ are collinear and perpendicular to $\Phi_i^k(P)$, with q^+ to the farther positive side. Since (n-1)D cells' normal vectors point to the polytope's exterior, then the normal vector of $\Phi_i^k(P)$ at $q \in FD_k^i(P)$ points towards q^+, i.e., to the positive side of the coordinate X_i-axis which is perpendicular to $\Phi_i^k(P)$. In a similar way, backward differences $\left(\pi_i\left(S_k^i(P)\right) -* \ \pi_i\left(S_{k-1}^i(P)\right)\right)$ correspond to the set of points q such that its corresponding points q^- and q^+ satisfy $q^- \notin S_{k-1}^i(P)$ and $q^+ \in S_k^i(P)$. Thus the normal vector of $\Phi_i^k(P)$ at $q \in BD_k^i(P)$ points towards q^-, i.e., to the negative side of the coordinate X_i-axis which is perpendicular to $\Phi_i^k(P)$. $\square$

Table 5.10 shows the extraction of volumes and their correct orientation through forward and backward differences. In this example we work with the 4D-OPP p shown in **Figure 5.6.2**. The first row shows sections perpendicular to X_1-axis. Through them we compute forward differences $FD_1^1(p)$ to $FD_4^1(p)$ in order to obtain the 3D-cells whose normal vector points towards the positive side of X_1-axis. On the other hand, these same sections share us to compute backward differences $BD_1^1(p)$ to $BD_4^1(p)$ which are composed by the set of 3D-cells whose normal vector points towards the negative side of X_1-axis. In a similar way, the remaining rows of **Table 5.10** shows sections perpendicular to X_2, X_3 and X_4-axes and the way their respective forward and backward differences are computed.

5.5.5. Virtual Couplets

Definition 5.22 [Aguilera98]: *An empty couplet will be called <u>virtual couplet</u>. Let p be an nD-OPP. We will say that p has a virtual couplet* $\Phi_k^i(p)$ *perpendicular to X_i-axis if no vertex of p lies on* $H(\Phi_k^i(p))$.

It must be noted that $S_{k-1}^i(p)$ and $S_k^i(p)$ in **Theorem 5.16** must be two consecutive sections of an nD-OPP p, but they (and thus, their projections) must not necessarily be different. Therefore if $\pi_i(S_{k-1}^i(p)) = \pi_i(S_k^i(p))$ then, by **Theorem 5.16**, $\Phi_k^i(p) = \varnothing$. This means that any number of virtual couplets may be considered wherever they are needed, without altering p.

5.6. Regularized Boolean Operations on the nD-EVM

5.6.1. Unifying Operands' Grids

Definition 5.23: *Let p be an nD-OPP.* $\underline{Spl_i(p) = \left\{H\left(\Phi_1^i(p)\right), H\left(\Phi_2^i(p)\right), ..., H\left(\Phi_{np_i}^i(p)\right)\right\}}$ *will be the ordered sequence of the supporting (n-1)D hyperplanes, perpendicular to X_i-axis, where the couplets* $\Phi_1^i(p)$, $\Phi_2^i(p)$,..., $\Phi_{np_i}^i(p)$ *lie.*

Definition 5.24: *Let p and q be two nD-OPP's. Let* $\underline{Spl_i(p \mid q)}$ *be the ordered sequence of supporting (n-1)D hyperplanes from p extended with those from q, defined as the ordered merge of* $Spl_i(p)$ *and* $Spl_i(q)$. *Let* $\underline{n_{i,pq}}$ *be the number of elements in* $Spl_i(p \mid q)$. *An (n-1)D hyperplane in the sequence will be labeled as* $\underline{H_k^i(p \mid q)}$, $k \in [1, n_{i,pq}]$.

Property 5.11: *Let p and q be two nD-OPP's and r = p op* q where op* is in* $\{\cup*, \cap*, -*, \otimes*\}$. *The following relations hold:*

- $Spl_i(p \mid q) = Spl_i(q \mid p)$
- $Spl_i(p), Spl_i(q), Spl_i(r) \subseteq Spl_i(p \mid q)$

Definition 5.25: *Let p and b be two nD-OPP's and r = p op* q where op* is in* $\{\cup^*, \cap^*, -^*, \otimes^*\}$. *Let* $\underline{part_i(p \mid q)}$ *be the* <u>*Sequential Spatial Partition*</u> *of* $\mathbb{R}^n$ *by means of* $Spl_i(p \mid q)$ *into* $n_{i,pq}+1$ *regions (*$region_0^i$ *to* $region_{n_{i,pq}}^i$*). Let* $\underline{Slice_k^i(p \mid q)}$ *and* $\underline{S_k^i(p \mid q)}$ *as the k-th slice and section of p inside* $region_k^i$ *respectively (Note that* $Slice_k^i(p \mid q) \neq Slice_k^i(q \mid p)$ *because the first one belongs to p while the second one belongs to q). The corresponding slice of the result r will be specified as* $\underline{Slice_k^i(r \mid p \mid q)}$ *which also can be written as* $Slice_k^i(r \mid q \mid p)$. *Let* $\underline{\Phi_k^i(p \mid q)}$ *be the corresponding couplet of p lying on* $H_k^i(p \mid q)$.

Figure 5.21 shows two 2D-OPP's p (in white color) and q (in light gray color). The intersection between p and q is shown in dark gray. By definition $Spl_1(p \mid q) = \{ H_1^1(p \mid q), H_2^1(p \mid q), H_5^1(p \mid q), H_3^1(p \mid q), H_4^1(p \mid q), H_6^1(p \mid q) \}$ where $H_1^1(p \mid q)$, $H_2^1(p \mid q)$ & $H_5^1(p \mid q)$ belong to p, and $H_3^1(p \mid q)$, $H_4^1(p \mid q)$ & $H_6^1(p \mid q)$ belong to q. $Spl_1(p \mid q)$ induces the partition of the 2D space in 7 regions, namely $region_0^1$ to $region_6^1$. And finally, $Slice_1^1(p \mid q)$ to $Slice_5^1(p \mid q)$ are the result of the partition of p induced by $Spl_1(p \mid q)$ ($Slice_5^1(p \mid q) = \varnothing$). Similarly, $Slice_1^1(q \mid p)$ to $Slice_5^1(q \mid p)$ are the result of the partition of q induced by $Spl_1(p \mid q)$ ($Slice_1^1(q \mid p) = Slice_2^1(q \mid p) = \varnothing$).

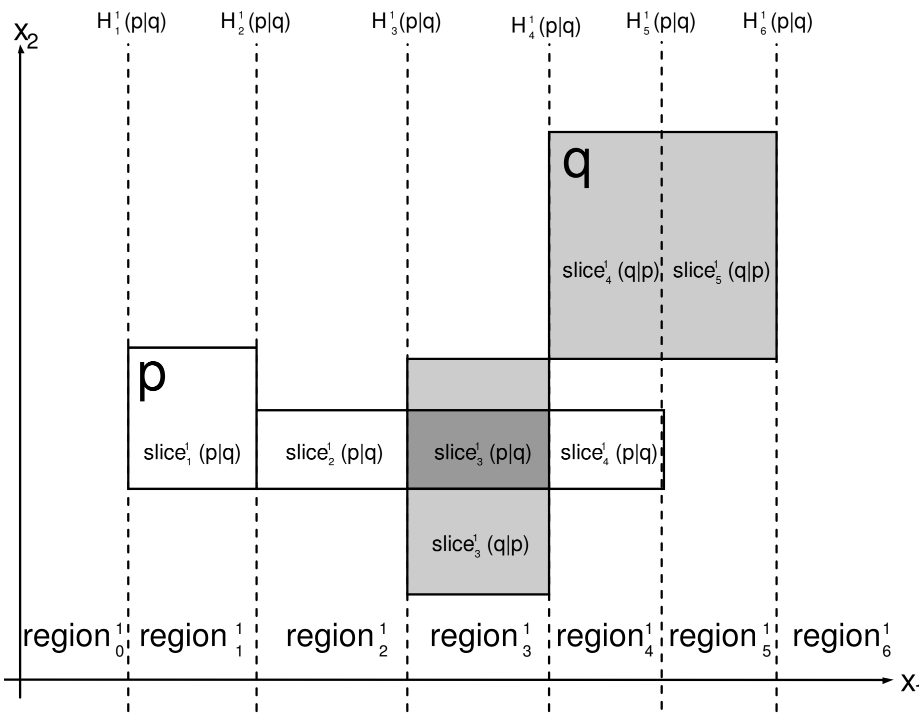


Figure 5.21. The partition of 2D space and of the 2D-OPP's p and q induced by $Spl_1(p \mid q)$ (see text for details).

Lemma 5.10: *Let p and b be two nD-OPP's and r = p op* q where op* is in* $\{\cup^*, \cap^*, -^*, \otimes^*\}$. *For each* $region_k^i$

$$Slice_k^i(r \mid p \mid q) = Slice_k^i(p \mid q) \;\; op^* \;\; Slice_k^i(q \mid p)$$

Proof [Aguilera98]:
Since r = p op* q and we are applying the same partition $part_i(p \mid q)$ to all three nD-OPP's p, q and r, then the proposition is proved by applying the procedures described for the Hyperspatial Partitioning Representations (see **Section 2.2. Figure 5.21** also shows an example). ◪

Corollary 5.6: *Let p and b be two nD-OPP's and r = p op* q where op* is in* $\{\cup^*, \cap^*, -^*, \otimes^*\}$. *For each* $region_k^i$,

$$\pi_i\left(S_k^i(r \mid p \mid q)\right) = \pi_i\left(S_k^i(p \mid q)\right) \;\; op^* \;\; \pi_i\left(S_k^i(q \mid p)\right)$$

Moreover, if all these sections lie in the same (n-1)D hyperplane (within $region_k^i$*), then*

$$S_k^i(r \mid p \mid q) = S_k^i(p \mid q) \;\; op^* \;\; S_k^i(q \mid p)$$

Proof [Aguilera98]:
The proof comes directly from **Lemma 5.10** and from the fact that $S_k^i(r \mid p \mid q)$, $S_k^i(p \mid q)$ and $S_k^i(q \mid p)$ are the sections of $Slice_k^i(r \mid p \mid q)$, $Slice_k^i(p \mid q)$ and $Slice_k^i(q \mid p)$ respectively. The corresponding projections need to be used only if those sections do not lie on the same (n-1)D hyperplane. ◪

From now on, $\Phi_k^i(p)$, $\Phi_k^i(q)$ and $\Phi_k^i(r)$ are to be written instead of $\Phi_k^i(p \mid q)$, $\Phi_k^i(q \mid p)$ and $\Phi_k^i(r \mid p \mid q)$ respectively, even though p, q and r are involved as in r = p op* q. It is to be understood, however that partition $part_i(p \mid q)$ must be applied to them, with the intrinsic addition of some virtual couplets to them.

5.6.2. The Regularized XOR operation on the nD-EVM

This section shows that the Regularized Exclusive OR (XOR) operation, $\otimes^*$, over two nD-OPP's, p and q, can be easily carried out using the nD-EVM. Let p, q and r be nD-OPP's such that r = p $\otimes^*$ q. Specifically, it will be shown that

$$EVM_n(p \otimes^* q) = EVM_n(p) \otimes EVM_n(q)$$

We will reproduce the proof given in [Aguilera98] which is by induction over the number of dimensions. We will describe the 1D case first, which in turn is subdivided into the single shell case (**Lemma 5.11**) and the general multiple shell case (**Lemma 5.12**), both Lemmas were also proved in [Aguilera98]. Finally the inductive case is presented in **Theorem 5.19**. It should be noted that while nD-OPP's p, q and r are infinite sets of points, while $EVM_n(p)$, $EVM_n(q)$ and $EVM_n(p \otimes^* r)$ are finite sets of points, because they only contain the Extreme Vertices of p, q and p $\otimes^*$ q respectively.

Lemma 5.11 [Aguilera98]: *Let* $p = \overline{ab}$ *and* $q = \overline{cd}$ *be two 1D-OPP's, each consisting of only one shell (i.e., one segment) and having* $EVM_1(p) = \{a,b\}$ *and* $EVM_1(q) = \{c,d\}$ *as their respective Extreme Vertices Models in the 1D space, with* $a,b,c,d \in \mathbb{R}$ *and a < b and c < d. Then*

$$EVM_1(p \otimes^* q) = EVM_1(p) \otimes EVM_1(q)$$

Proof:
Two collinear segments p and q can be disjoint, contiguous, totally coincident, one contained in other, or partially overlapping to each other. Then p $\otimes^*$ q will be a 1D-OPP consisting of zero, one or two shells (i.e., segments). The proposition is proved by exhaustive characterization of all possible cases, then applying the regularized formula $p \otimes^* q = (p \cup^* q) -^* (p \cap^* q)$. These cases are shown in **Table 5.11**, showing that all these cases hold that

$$EVM_1(p \otimes^* q) = \{a,b\} \otimes \{c,d\} = EVM_1(p) \otimes EVM_1(q)$$

◪

Lemma 5.12 [Aguilera98]: *Let p and q be two 1D-OPP's of any number of shells, having $EVM_1(p)$ and $EVM_1(q)$ as their respective models, then*

$$EVM_1(p \otimes^* q) = EVM_1(p) \otimes EVM_1(q)$$

Proof:
Let $p = \bigcup_i p_i$, $q = \bigcup_j q_j$ and $r = p \otimes^* q = \bigcup_k r_k$ where $p_i = \overline{a_i b_i}$, $q_j = \overline{c_j d_j}$ and $r_k = \overline{e_k f_k}$ are all the shells of p, q and r respectively, with $a_i < b_i < a_{i+1} < b_{i+1}$, similarly $c_j < d_j < c_{j+1} < d_{j+1}$ and $e_k < f_k < e_{k+1} < f_{k+1}$ where $a_i, b_i, c_j, d_j, e_k, f_k \in \mathbb{R},\ \forall i, j, k$. Then $EVM_1(p) = \{a_1, b_1, a_2, b_2, ...\}$, $EVM_1(q) = \{c_1, d_1, c_2, d_2, ...\}$ and $EVM_1(r) = \{e_1, f_1, e_2, f_2, ...\}$.

A point $x \in EVM_1(r) = EVM_1(p \otimes^* q)$ is one of the two Extreme Vertices of a segment $r_k \subset r$, that is, $x = e_k$ or $x = f_k$ for some k. On the other hand, each segment in r is the result of the XOR operation between segments from p and/or q as shown in **Lemma 5.11**. Thus, each Extreme Vertex of any segment r_k must also be an Extreme Vertex of either p or q, that is, $x \in EVM_1(p)$ or $x \in EVM_1(q)$. Therefore, $EVM_1(p \otimes^* q) \subseteq EVM_1(p) \cup EVM_1(q)$.

Furthermore, $x \in EVM_1(p) \cap EVM_1(q) \Leftrightarrow x \notin EVM_1(p \otimes^* q)$, as shown in the cases 2.1, 2.2, 3 and 4.1 to 4.6 from **Lemma 5.11** (**Table 5.11**). Thus $EVM_1(p \otimes^* q) = \left(EVM_1(p) \cup EVM_1(q)\right) - \left(EVM_1(p) \cap EVM_1(q)\right)$.

Therefore, $EVM_1(p \otimes^* q) = EVM_1(p) \otimes EVM_1(q)$. ◪

Case:	If $\overline{ab}$ & $\overline{cd}$ are:	$\{a,b\}\otimes\{c,d\} =$ $EVM_1(p)\otimes EVM_1(q)$	$p\cup^* q$	$p\cap^* q$	$p\otimes^* q =$ $(p\cup^* q)-^*(p\cap^* q)$	$EVM_1(\overline{ab}\otimes^* \overline{cd})$
1.1	Disjoint: b < c	{a, b, c, d}	$\overline{ab}\cup\overline{cd}$	$\varnothing$	$\overline{ab}\cup\overline{cd}$	{a, b, c, d}
1.2	Disjoint d < a	{c, d, a, b}	$\overline{ab}\cup\overline{cd}$	$\varnothing$	$\overline{ab}\cup\overline{cd}$	{c, d, a, b}
2.1	Contiguous: b = c	{a, d}	$\overline{ad}$	$\varnothing$	$\overline{ad}$	{a, d}
2.2	Contiguous: a = d	{c, b}	$\overline{cb}$	$\varnothing$	$\overline{cb}$	{c, b}
3	Coincident: a = c and b = d	$\varnothing$	$\overline{ab}\equiv\overline{cd}$	$\overline{ab}\equiv\overline{cd}$	$\varnothing$	$\varnothing$
4.1 4.2 4.3	Inclusive ($\overline{ab}\supset\overline{cd}$): a < c < d < b a = c < d < b a < c < d = b	{a, c, d, b} {d, b} {a, c}	$\overline{ab}$ $\overline{ab}\equiv\overline{cb}$ $\overline{ab}\equiv\overline{ad}$	$\overline{cd}$ $\overline{ad}\equiv\overline{cd}$ $\overline{cb}\equiv\overline{cd}$	$\overline{ac}\cup\overline{db}$ $\overline{db}$ $\overline{ac}$	{a, c, d, b} {d, b} {a, c}
4.4 4.5 4.6	Inclusive ($\overline{ab}\subset\overline{cd}$): c < a < b < d c = a < b < d c < a < b = d	{c, a, b, d} {b, d} {c, a}	$\overline{cd}$ $\overline{ad}\equiv\overline{cd}$ $\overline{cb}\equiv\overline{cd}$	$\overline{ab}$ $\overline{ab}\equiv\overline{cb}$ $\overline{ab}\equiv\overline{ad}$	$\overline{ca}\cup\overline{bd}$ $\overline{bd}$ $\overline{ca}$	{c, a, b, d} {b, d} {c, a}
5.1	Overlapping: a < c < b < d	{a, c, b, d}	$\overline{ad}$	$\overline{cb}$	$\overline{ac}\cup\overline{bd}$	{a, c, b, d}
5.2	Overlapping: c < a < d < b	{c, a, d, b}	$\overline{cb}$	$\overline{ad}$	$\overline{ca}\cup\overline{db}$	{c, a, d, b}

Table 5.11. Case analysis for computing $EVM_1(p\otimes^* q)$ of two segments.

Before going any further, according to **Lemma 5.5** we have that $EVM_{n-1}(\pi_j(\Phi_k^j(p))) = \pi_j(EV_k^j(p))$ where $EV_k^j(p)$ is the set of Extreme Vertices that lies on the supporting hyperplane $H\left(\Phi_k^j(p)\right)$ of the couplet $\Phi_k^j(p)$. It is easy to observe that $EVM_{n-1}(\pi_j(\Phi_k^j(p)))$ and $EV_k^j(p)$ have exactly the same vertices except by the fact that vertices in

$EV_k^j(p)$ share the additional same x_j coordinate. Hence, $EVM_{n-1}(\pi_j(\Phi_k^j(p))) = EV_k^j(p)$ by adding to the vertices in $EVM_{n-1}(\pi_j(\Phi_k^j(p)))$ the corresponding x_j coordinate which is common to the vertices in $EV_k^j(p)$. Starting from this point, when we refer to $EVM_n(\Phi_k^j(p))$ we denote to the set of extreme vertices according to the following

Definition 5.26: $\underline{EVM_n(\Phi_k^j(p))}$ *denotes to the extreme vertices in* $EVM_{n-1}(\pi_j(\Phi_k^j(p)))$ *with an additional common x_j coordinate which has been taken from the vertices in* $EV_k^j(p)$.

Theorem 5.19 [Aguilera98]: *Let p and q be two nD-OPP's having* $EVM_n(p)$ *and* $EVM_n(q)$ *as their respective Extreme Vertices Models in nD space, then*

$$EVM_n(p \otimes^* q) = EVM_n(p) \otimes EVM_n(q)$$

Proof:
The approach to prove the proposition is by induction over the number of dimensions. The 1D case has already been proved in **Lemmas 5.11** and **5.12** for the single shell case and for the general multiple shell case respectively. They represent the base cases.

For the inductive case, let $r = p \otimes^* q$ be the resulting nD-OPP and assume that the same partition $part_i(p \mid q)$ has been applied to p, q and r, with the intrinsic addition of some virtual couplets. Finally, let us suppose that the relation $EVM_n(p \otimes^* q) = EVM_n(p) \otimes EVM_n(q)$ holds for (n-1)D space. Then:

$EVM_n(p \otimes^* q) = EVM_n(r) = \bigcup_k EV_k^i(r) = \bigcup_k EVM_n(\Phi_k^i(r))$ (by **Property 5.5** and **Definition 5.26**)

$\Rightarrow EVM_n(r) = \bigotimes_k EVM_n\left(\Phi_k^i(r)\right)$ (since every $\Phi_k^i(p)$ lies on distinct (n-1)D hyperplanes)

For each k,

$\pi_i\left(\Phi_k^i(r)\right) = \pi_i\left(S_{k-1}^i(r)\right) \otimes^* \pi_i\left(S_k^i(r)\right)$ (by **Theorem 5.16**)

$\quad = \pi_i\left(S_{k-1}^i(p) \otimes^* S_{k-1}^i(q)\right) \otimes^* \pi_i\left(S_k^i(p) \otimes^* S_k^i(q)\right)$ (by **Corollary 5.6**)

$\quad = \left(\pi_i\left(S_{k-1}^i(p)\right) \otimes^* \pi_i\left(S_k^i(p)\right)\right) \otimes^* \left(\pi_i\left(S_{k-1}^i(q)\right) \otimes^* \pi_i\left(S_k^i(q)\right)\right)$ (by associativity)

Therefore, $\pi_i\left(\Phi_k^i(r)\right) = \pi_i\left(\Phi_k^i(p)\right) \otimes^* \pi_i\left(\Phi_k^i(q)\right)$ (by **Theorem 5.16**)

Moreover, $\Phi_k^i(r) = \Phi_k^i(p) \otimes^* \Phi_k^i(q)$ (since all three lie in the same (n-1)D space, thus they do not need to be projected)

Furthermore, $\Phi_k^i(p)$, $\Phi_k^i(q)$ and $\Phi_k^i(r)$ are all (n-1)D-OPP's, thus, by applying the inductive hypothesis

$$EVM_{n-1}(\Phi_k^i(r)) = EVM_{n-1}(\Phi_k^i(p)) \otimes EVM_{n-1}(\Phi_k^i(q))$$

Also observe that regularized XOR is nor longer needed, since now these are finite and discrete sets of points.

Now, by considering that $EVM_n(r) = \bigotimes_k EVM_n\left(\Phi_k^i(r)\right)$ at the beginning of this proof, we have

$\bigotimes_k EVM_n\left(\Phi_k^i(r)\right) = \bigotimes_k \left[EVM_n\left(\Phi_k^i(p)\right) \otimes EVM_n\left(\Phi_k^i(q)\right)\right]$ (By applying **Definition 5.26**)

$\quad = \left[\bigotimes_k EVM_n\left(\Phi_k^i(p)\right)\right] \otimes \left[\bigotimes_k EVM_n\left(\Phi_k^i(q)\right)\right] = EVM_n(p) \otimes EVM_n(q)$ (by **Property 5.5**)

$\therefore\ EVM_n(p \otimes^* q) = EVM_n(p) \otimes EVM_n(q)$ ◪

This result allows expressing a formula for computing nD-OPP's sections from couplets and vice-versa, by means of their corresponding Extreme Vertices Models. These formulae are obtained by combining **Theorem 5.19** with **Theorem 5.16**; and **Theorem 5.19** with **Theorem 5.17**, respectively:

Corollary 5.7 [Aguilera98]: $EVM_{n-1}\left(\pi_i(\Phi_k^i(p))\right) = EVM_{n-1}\left(\pi_i(S_{k-1}^i(p))\right) \otimes EVM_{n-1}\left(\pi_i(S_k^i(p))\right)$ ◪

Corollary 5.8 [Aguilera98]: $EVM_{n-1}\left(\pi_i(S_k^i(p))\right) = EVM_{n-1}\left(\pi_i(S_{k-1}^i(p))\right) \otimes EVM_{n-1}\left(\pi_i(\Phi_k^i(p))\right)$ ◪

Finally, the following two corollaries can be stated, which correspond to specific situations of the XOR operands. They allow computing, respectively, the union and difference of two nD-OPP's when those specific situations are met.

Corollary 5.9 [Aguilera98]: *Let p and q be two disjoint or quasi disjoint nD-OPP's having $EVM_n(p)$ and $EVM_n(q)$ as their respective Extreme Vertices Models, then* $EVM_n(p \cup q) = EVM_n(p) \otimes EVM_n(q)$.

Proof:
If p and q are two disjoint or quasi disjoint nD-OPP's then $p \cap^* q = \varnothing$, therefore $p \otimes^* q = (p \cup q) - \varnothing = p \cup q$, thus, by **Theorem 5.19**, $EVM_n(p \cup q) = EVM_n(p) \otimes EVM_n(q)$. ◪

Corollary 5.10 [Aguilera98]: *Let p and q be nD-OPP's such that* $p \supseteq q$ *having $EVM_n(p)$ and $EVM_n(q)$ as their respective Extreme Vertices Models, then* $EVM_n(p -^* q) = EVM_n(p) \otimes EVM_n(q)$ *(the complement of q with respect to p).*

Proof:
If $p \supseteq q$ then $q -^* p = \varnothing$, thus $p \otimes q = (p -^* q) \cup \varnothing = p -^* q$; then by **Theorem 5.19**, $EVM_n(p -^* q) = EVM_n(p) \otimes EVM_n(q)$. ◪

5.6.3. The Regularized Union, Intersection and Difference operations on the nD-EVM

Before to proceed to the Boolean operations main theorem let's to describe the way a Boolean operation should be performed recursively between two nD-OPP's by taking in account the relations between their respective sections. In order to express the main idea behind regularized Boolean operations under the nD-EVM lets to consider an example. Let A and B the two 4D-OPP's operands of the **Table 5.12**. The 4D-OPP A can be seen as a four-dimensional "*cross-shaped*" polytope and the 4D-OPP B can be considered as a four-dimensional "*L-shaped*" polytope (see **Table 5.12**, first column). The operand A has three sections while operand B has only two (see **Table 5.12**, second column). Each 3D section will have only one 2D section (since they are only rectangular prisms; third column). Finally, each 2D section will have only one 1D section: a segment with their respective pair of extreme vertices (fourth column). The 1D sections' extreme vertices for operand A are labeled as a_i and b_i while the 1D sections' extreme vertices for operand B are labeled as c_i and d_i. At this point it is important to consider that 3D sections, 2D sections and 1D sections are projected, and thus embedded, into 3D, 2D and 1D spaces respectively, in order to appropriately have access to their corresponding EVM's.

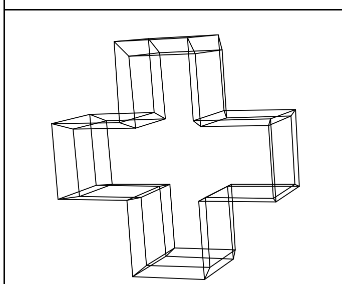
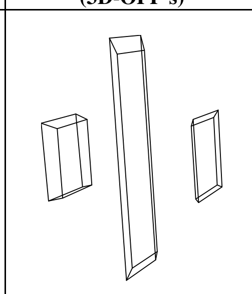
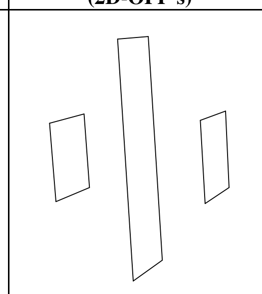
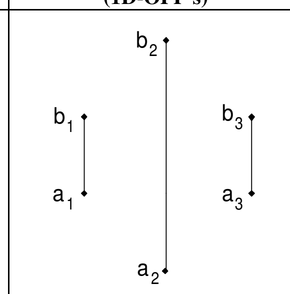
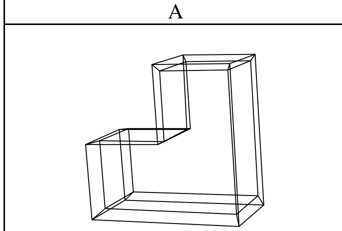
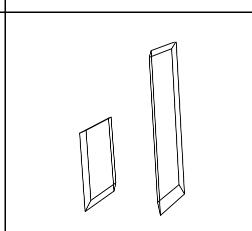
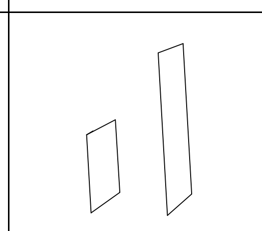
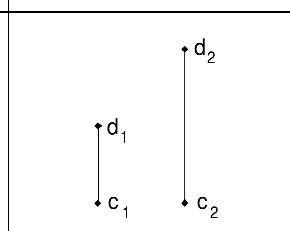
4D-OPP's	Sections (3D-OPP's)	Sections (2D-OPP's)	Sections (1D-OPP's)
 A			
 B			

Table 5.12. Two 4D-OPP's A & B and their corresponding sections since the 3D case until the 1D case (see text for details).

The relative position for the Boolean operation is shown in **Figure 5.22.a** (the Boolean operation between the two 4D-OPP's). In the **Figure 5.22.b** is shown how interact the 3D sections for operands A and B (the Boolean operation between the 3D sections). In **Figure 5.22.c** are shown the interactions between the 2D sections (the

Boolean operation between the 2D sections). Finally, in **Figure 5.22.d** are shown the interactions between the 1D sections (the basic case for the Boolean operations).

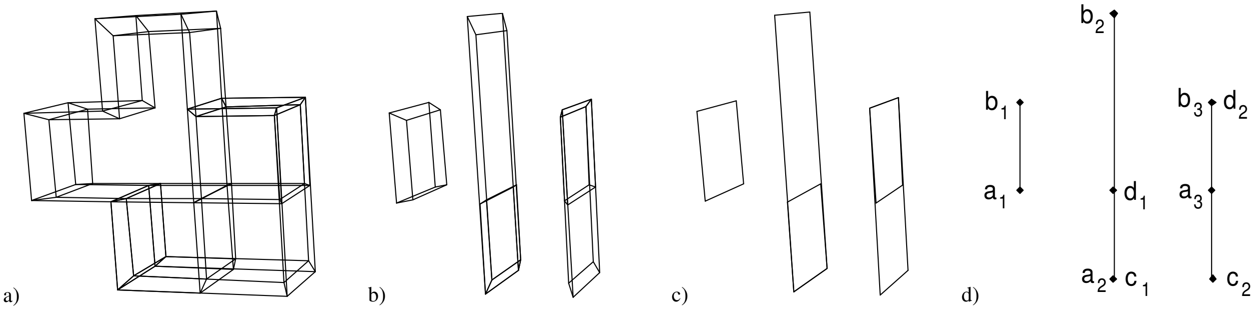


Figure 5.22. Two 4D-OPP's (presented in **Table 5.12**) with common interior regions (a).
b) Their 3D sections (two of them have common interiors). c) The 2D sections from the 3D sections. d) The 1D sections from the 2D sections.

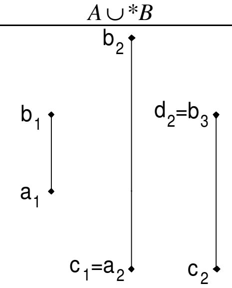
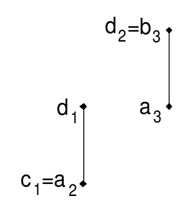
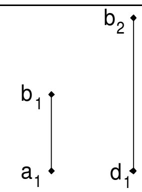
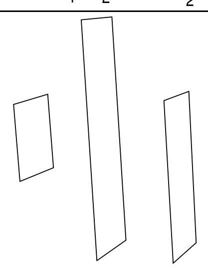
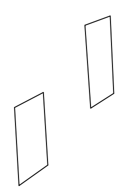
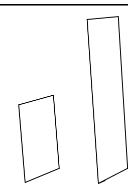
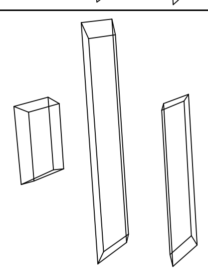
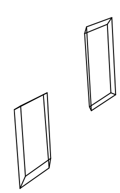
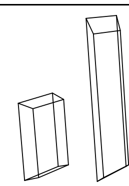
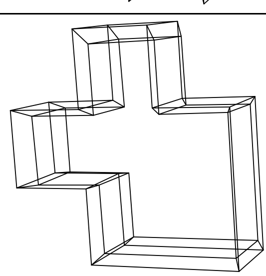
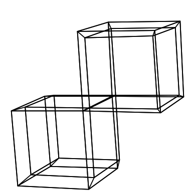
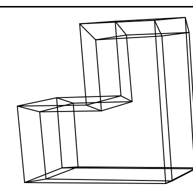
$A \cup^* B$	$A \cap^* B$	$A -^* B$
		
		
		
		

Table 5.13. Boolean Operations Between 1D Sections of two 4D-OPP's
(whose relative positions are shown in **Figure 5.22.a)** and the resultant 4D-OPP's (see text for details).

Since the segments in **Figure 5.22.d** represent the basic case for the regularized Boolean operations between the 4D-OPP's A and B (of **Figure 5.22.a**), it must be applied the corresponding operator. We will exemplify the operations of union, intersection and difference. In the **Table 5.13** are shown the results of these operations. **Table 5.13**, columns 1, 2 and 3, correspond to $A \cup^* B$, $A \cap^* B$ and $A -^* B$ respectively. The Boolean operations between 1D sections are performed through a straightforward method. The resultant 1D sections will define 2D rectangular sections, through **Theorems 5.16** and **5.17**, which in turn define, again through **Theorems 5.16** and **5.17**, the three or two (according to the Boolean operation) 3D sections of the resultant 4D-OPP.

If $\overline{ab}$ & $\overline{cd}$ are:	$EVM_1(\overline{ab} \cup^* \overline{cd})$	$EVM_1(\overline{ab} \cap^* \overline{cd})$	$EVM_1(\overline{ab} -^* \overline{cd})$	$EVM_1(\overline{ab} \otimes^* \overline{cd})$
Disjoint: b < c	{a, b, c, d}	∅	{a, b}	{a, b, c, d}
Disjoint d < a	{c, d, a, b}	∅	{a, b}	{c, d, a, b}
Contiguous: b = c	{a, d}	∅	{a, b}	{a, d}
Contiguous: a = d	{c, b}	∅	{a, b}	{c, b}
Coincident: a = c and b = d	{a = c, b = d}	{a = c, b = d}	∅	∅
Inclusive ($\overline{ab} \supset \overline{cd}$): a < c < d < b a = c < d < b a < c < d = b	{a, b} {a = c, b} {a, d = b}	{c, d} {a = c, d} {c, d = b}	{a, c, d, b} {d, b} {a, c}	{a, c, d, b} {d, b} {a, c}
Inclusive ($\overline{ab} \subset \overline{cd}$): c < a < b < d c = a < b < d c < a < b = d	{c, d} {c = a, d} {c, b = d}	{a, b} {c = a, b} {a, b = d}	∅ ∅ ∅	{c, a, b, d} {b, d} {c, a}
Overlapping: a < c < b < d	{a, d}	{c, b}	{a, c}	{a, c, b, d}
Overlapping: c < a < d < b	{c, b}	{a, d}	{d, b}	{c, a, d, b}

Table 5.14. The Boolean regularized operations between two 1D-OPP's under the 1D-EVM and their possible cases.

Now we present the following

Theorem 5.20 [Aguilera98]: *A regularized Boolean operation, op*, where op** $\in \{\cup^*, \cap^*, -^*, \otimes^*\}$, *over two nD-OPP's p and q, both expressed in the nD-EVM, can be carried out by means of the same op* applied over their own sections, expressed through their Extreme Vertices Models, which are (n-1)D-OPP's.*

Proof:

By applying the same partition $part_i(p \mid q)$ to p and q, let r = p op* q, we have:

$$\pi_i\left(EVM_n(p\ op^*\ q)\right) = \pi_i\left(EVM_n(r)\right) = \bigcup_k EVM_{n-1}\left(\pi_i\left(\Phi_k^i(r)\right)\right) \quad \text{(by **Theorem 5.13**)}$$

$$= \bigcup_k \left[EVM_{n-1}\left(\pi_i\left(S_{k-1}^i(r)\right)\right) \otimes EVM_{n-1}\left(\pi_i\left(S_k^i(r)\right)\right)\right] \quad \text{(by **Corollary 5.7**)}$$

$$= \bigcup_k \begin{bmatrix} EVM_{n-1}\left(\pi_i\left(S_{k-1}^i(p)\right)\ op^*\ \pi_i\left(S_{k-1}^i(q)\right)\right) \otimes \\ EVM_{n-1}\left(\pi_i\left(S_k^i(p)\right)\ op^*\ \pi_i\left(S_k^i(q)\right)\right) \end{bmatrix} \quad \text{(by **Corollary 5.6**)}$$

$$= \bigcup_k \begin{bmatrix} EVM_{n-1}\left(S_{k-1}^i(p)\ op^*\ S_{k-1}^i(q)\right) \otimes \\ EVM_{n-1}\left(S_k^i(p)\ op^*\ S_k^i(q)\right) \end{bmatrix} \quad \text{(Because the sections lie in the same (n-1)D space we remove the projections)}$$

$$\Rightarrow \pi_i\left(EVM_n(p\ op^*\ q)\right) = \bigcup_k \left[EVM_{n-1}\left(S_{k-1}^i(p)\ op^*\ S_{k-1}^i(q)\right) \otimes EVM_{n-1}\left(S_k^i(p)\ op^*\ S_k^i(q)\right)\right]$$

Thus, for two nD-OPP's, p and q, $EVM_n(p\ op^*\ q)$ is expressed in terms of the same op* applied over their own sections, which are (n-1)D-OPP's. ◽

This result leads into a recursive process for computing the Regularized Boolean operations using the nD-EVM, which descends on the number of dimensions. The base or trivial case of the recursion is the 1D-Boolean operations which can be performed using direct methods (see **Table 5.14**).

Once each term in $\bigcup_k \left[EVM_{n-1}\left(S_{k-1}^i(p)\ op^*\ S_{k-1}^i(q)\right) \otimes EVM_{n-1}\left(S_k^i(p)\ op^*\ S_k^i(q)\right)\right]$ has been computed recursively, we apply **Corollary 5.7** in order to get $\bigcup_k EVM_{n-1}\left(\pi_i\left(\Phi_k^i(p\ op^*\ q)\right)\right)$. By applying **Definition 5.26** to each one of its terms we finally get $EVM_n(p\ op^*\ q) = \bigcup_k EVM_n\left(\Phi_k^i(p\ op^*\ q)\right)$.

The regularized XOR, as a Boolean operation, can also be carried out using the method described by **Theorem 5.20**, although there is a much simpler and faster method, as pointed out in **Theorem 5.19**.

5.7. nD-EVM Properties

In order to prove our Main Hypothesis we have to consider a metric that indicates that the EMV-nD is a complete representation scheme, i.e., that all the geometry, topology and correct boundary orientation of an nD-OPP can be unambiguously obtained from its EVM. In this aspect, Requicha defines a set of formal criterions that every scheme for representing solids must have rigorously defined [Requicha80]:

- **Domain:** The set of entities which are represented by the scheme. The domain's size must be enough to allow the representation of a useful set of objects, and therefore, it characterizes the scheme's power.
- **Completeness:** The representation can not be ambiguous. There are no doubts about what is represented. A representation must correspond to one and only one solid.
- **Uniqueness:** A representation is unique if it can be used to codify a certain solid in just one way.
- **Validity:** A representation scheme must disable the creation of an invalid representation, or in other words, a representation that does not correspond to a solid. Additionally, the object must keep the closure under rotation, translation and other operations. In this way, the operations between valid solids must return valid solids.

Although Requicha's formal criterions were defined for the Theory of Solid Modeling, we consider that they can be applied in our study under the context of Polytopes Modeling. We will call to the nD-EVM a complete scheme if it satisfies these four formal criterions.

In the following sections we will show how the Extreme Vertices Model in the n-Dimensional space, as a representation scheme for nD-OPP's, is characterized by the above formal properties: Domain, Validity, Completeness and Uniqueness.

5.7.1. Domain

The set of objects which are represented in the nD-EVM is clearly the complete set of n-Dimensional Orthogonal Pseudo-Polytopes.

5.7.2. Validity

The nD-EVM is cleary a very easy-to-validate set of points [Aguilera98]. This section presents a Theorem that provides a necessary and sufficient condition for a finite set of points to be a valid nD-EVM. Let $Q \subset \mathbb{R}^n$ be a set of points, then every point q $\in$ Q defines n possible orthogonal extended edges incident to q.

Definition 5.27 [Aguilera98]: *$Q \subset \mathbb{R}^n$ is an <u>all-even set of points</u> if and only if every possible extended edge, in each dimension of Q, holds an even number of points of Q.*

Theorem 5.21 [Aguilera98]: *Let Q be a finite set of points in $\mathbb{R}^n$, then*

Q is a valid nD-EVM for some n-Dimensional Orthogonal Pseudo-Polytope p, i.e., Q = $EVM_n(p)$

if and only if

Q is an all-even set of points.

Proof:

$\Rightarrow$)

By **Lemma 5.3** Card(EV_i(p)) is an even number for all i $\in$ {1,…, n}.

$\Leftarrow$)

We will prove the reciprocal by induction over the number of dimensions n.

For d = 1. If $Q \subset \mathbb{R}$ is an all-even set of points, i.e., Q has an even number of points q_1, q_2, … q_{2k} in the only possible extended edge, then the 1D-OPP p, composed by the k segments $\overline{q_1q_2},\ldots,\overline{q_{2k-1}q_{2k}}$ is a valid 1D-OPP holding that EVM_1(p) = Q.

Now, for the inductive step, let us assume the following:

- The Proposition is valid in $\mathbb{R}^{n-1}$, with n > 1, as our inductive hypothesis.
- Q is an all-even set of points in $\mathbb{R}^n$.
- $VX_i = \{a_1, a_2, \ldots, a_{np_i}\}$ is the set of all x_i-coordinate values of points in Q with $a_k < a_{k+1}$.
- Let $Q_k = \{q \in Q : q \cdot x_i = a_k,\ a_k \in VX_i\}$, i.e., $Q = \bigcup_{k=1}^{np_i} Q_k$, with $Q_j \cap Q_k = \varnothing$, for $j \neq k$.

The following statements that lead to the construction of p, so that EVM_n(p) = Q, can be deducted:

1) Since Q is an all-even set of points in $\mathbb{R}^n$ then Q_k is an all-even set of points in $\mathbb{R}^{n-1}$. Thus, Q_k is, by our inductive hypothesis, the EVM of some (n-1)D-OPP p_k with $Q_k = EVM_{n-1}(p_k)$.
2) If an nD-OPP p exists, such that EVM_n(p) = Q, then, for all k,
   - $\Phi_k^i(p) = p_k$ and therefore $EVM_n(\Phi_k^i(p)) = EVM_{n-1}(p_k) = Q_k$. Therefore, $\Phi_k^i(p)$ are valid (n-1)D-OPP's embedded in $\mathbb{R}^n$.
   - $S_0^i(p) = \varnothing$ is a valid OPP.
   - $\pi_i\left(S_k^i(p)\right) = \bigotimes_{j=1}^{k} {}^*\pi_i\left(\Phi_j^i(p)\right)$ by **Theorem 5.17**. Thus $S_k^i(p)$ are also valid (n-1)D-OPP's.

3) Finally, the last (and external) section $S^i_{np_i}(p)$ perpendicular to X_i-axis, which is supposed to be empty, and whose EVM is computed (according to **Theorem 5.17** and **Theorem 5.19**) as:

$$EVM_{n-1}\left(\pi_i\left(S^i_{np_i}(p)\right)\right) = \bigotimes_{k=1}^{np_i} EVM_{n-1}\left(\pi_i\left(\Phi^i_k(p)\right)\right)$$

will indeed be empty due to the fact that Q is an all-even set of points in $\mathbb{R}^n$. Therefore, p will be a valid nD-OPP with $EVM_n(p) = Q$. ◪

5.7.3. Completeness

The following Theorem proves that every valid nD-OPP representation on the nD-EVM is unambiguous.

Theorem 5.22 [Aguilera98]: *The Extreme Vertices Model in the n-Dimensional Space is a complete Boundary Representation for nD-OPP's.*
Proof:
To prove the Proposition we must show that all the geometry, topology and correct boundary orientation of a nD-OPP can be unambiguously obtained from its EVM.

Concerning geometry, from **Theorem 5.9**, all coordinate of non-extreme vertices appear as coordinate of the Extreme Vertices. Then, although non-extreme vertices do not appear in the nD-EVM, they can be inferred from this model. Concerning orientation and topology, according to **Theorem 5.18**, (n-1)D cells can be extracted with their correct orientation by using the forward and backward differences techniques. ◪

5.7.4. Uniqueness

By **Theorem 5.10** and by the fact we are considering only one ordering for the coordinates of the Extreme Vertices we conclude that for an nD-OPP p its $EVM_n(p)$ is unique.

5.8. Conclusions

We have now the elements to conclude that our initial and main claim is true:

Theorem 5.23:

The Extreme Vertices Model in the n-Dimensional Space (nD-EVM) is a complete scheme for the representation of n-Dimensional Orthogonal Pseudo-Polytopes.

Proof:

There are satisfied the following properties:

- Domain: See **Section 5.7.1**.
- Validity: By **Theorem 5.21**.
- Completeness: By **Theorem 5.22**.
- Uniqueness: See **Section 5.7.4**. ◪

5.8.1. Putting the nD-EVM Concepts Together: An Example

We will describe an example where all the concepts related to the nD-EVM are considered together. The objective is to show how starting from a valid EVM we get a boundary representation for a given polytope. In our case we exemplify by considering an all-even set in four dimensional space. In **Appendix G**, the reader can find explicit details about the specific coordinates of the considered set. Let p be the 4D-OPP to be obtained. **Tables 5.15** to **5.19** show each one of the steps behind the conversion EVM-Boundary Representation.

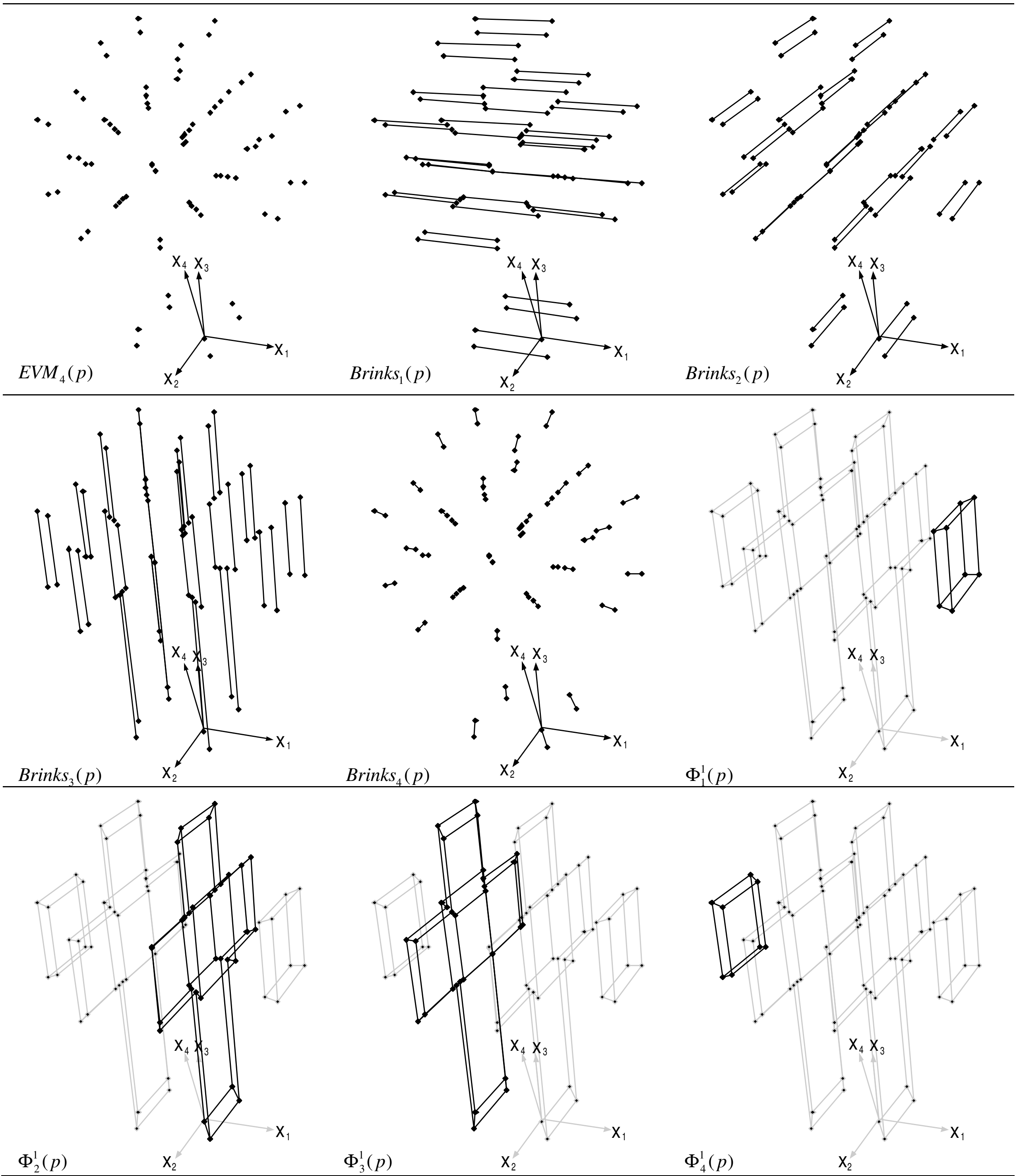


Table 5.15. Determining a 4D-OPP p from its 4D-EVM (Part 1, See text for details).

The considered all-even set is presented in **Table 5.15** and it is a valid 4D-EVM for p. We have that Card(EVM_4(p)) = 80. Brinks parallel to X_1, X_2, X_3, and X_4-axes can be found by matching up contiguous vertices on the appropriate extended edges ($Brinks_1$(p), $Brinks_2$(p), $Brinks_3$(p) and $Brinks_4$(p)). Couplets perpendicular to X_1-axis $\Phi_1^1(p)$, $\Phi_2^1(p)$, $\Phi_3^1(p)$ and $\Phi_4^1(p)$ can be found by linking $Brinks_2$(p), $Brinks_3$(p) and $Brinks_4$(p) together. In **Appendix G** can be seen the coordinates of the Extreme Vertices associated to each set in **Table 5.15**.

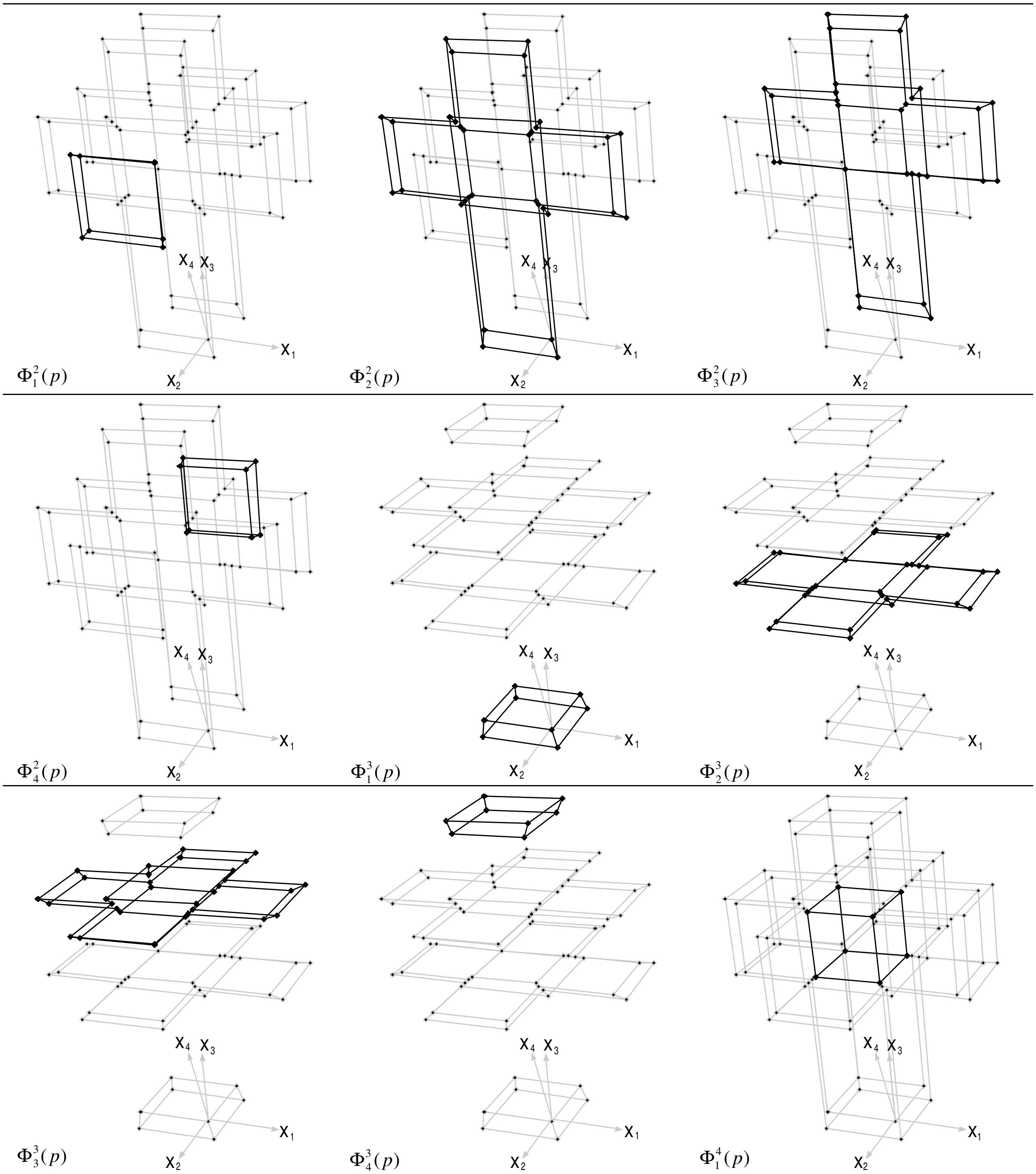


Table 5.16. Determining a 4D-OPP p from its 4D-EVM (Part 2, See text for details).

Couplets, in **Table 5.16**, perpendicular to X_2-axis $\Phi_1^2(p)$ to $\Phi_4^2(p)$ can be found by linking $Brinks_1(p)$, $Brinks_3(p)$ and $Brinks_4(p)$ together. Couplets perpendicular to X_3-axis ($\Phi_1^3(p)$ to $\Phi_4^3(p)$) and X_4-axis ($\Phi_1^4(p)$ and in **Table 5.17** $\Phi_2^4(p)$, $\Phi_3^4(p)$ and $\Phi_4^4(p)$) are obtained in similar way. Because of the obtained couplets we can be infer some symmetries to be present in the final 4D polytope. The 3D-EVM's for each $\Phi_1^i(p)$ and $\Phi_4^i(p)$, for all $1 \le i \le 4$, correspond to 3D boxes while the 3D-EVM's for each $\Phi_2^i(p)$ and $\Phi_3^i(p)$, for all $1 \le i \le 4$, correspond to 3D "crosses" not completely solid in the sense shown in **Figure 5.23.b**.

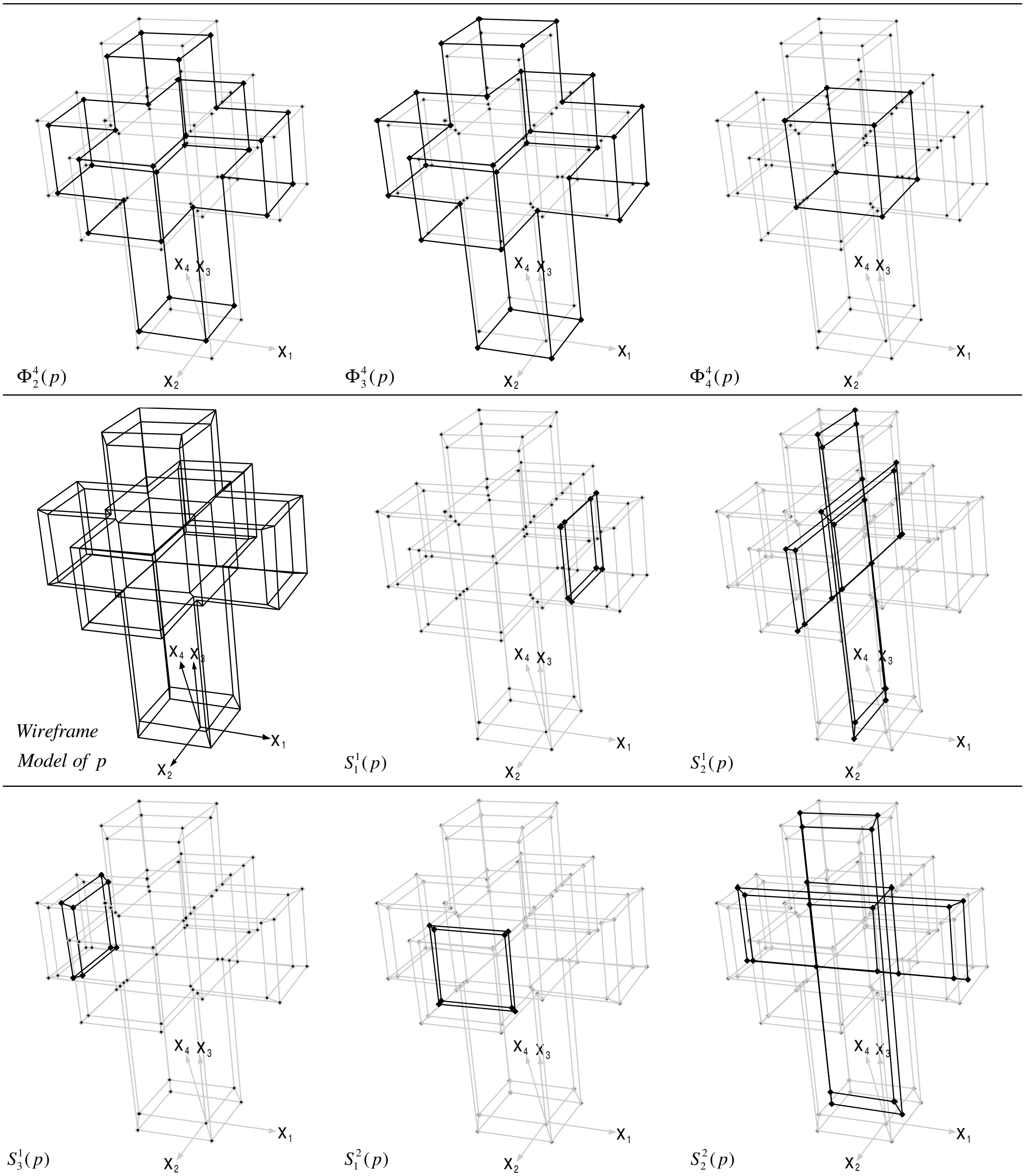


Table 5.17. Determining a 4D-OPP p from its 4D-EVM (Part 3, See text for details).

Table 5.17 shows how linking all brinks together leads to a wireframe model of the 4D-OPP p represented. By applying **Corollary 5.8** we obtain internal Sections $S_1^1(p)$ to $S_3^1(p)$ perpendicular to X_1-axis through $EVM_3\left(\pi_1(S_k^1(p))\right) = EVM_3\left(\pi_1(S_{k-1}^1(p))\right) \otimes EVM_3\left(\pi_1(\Phi_k^1(p))\right)$ for all $1 \leq k \leq 3$. **Appendix G** shows the coordinates of the Extreme Vertices associated to each set in **Table 5.17**.

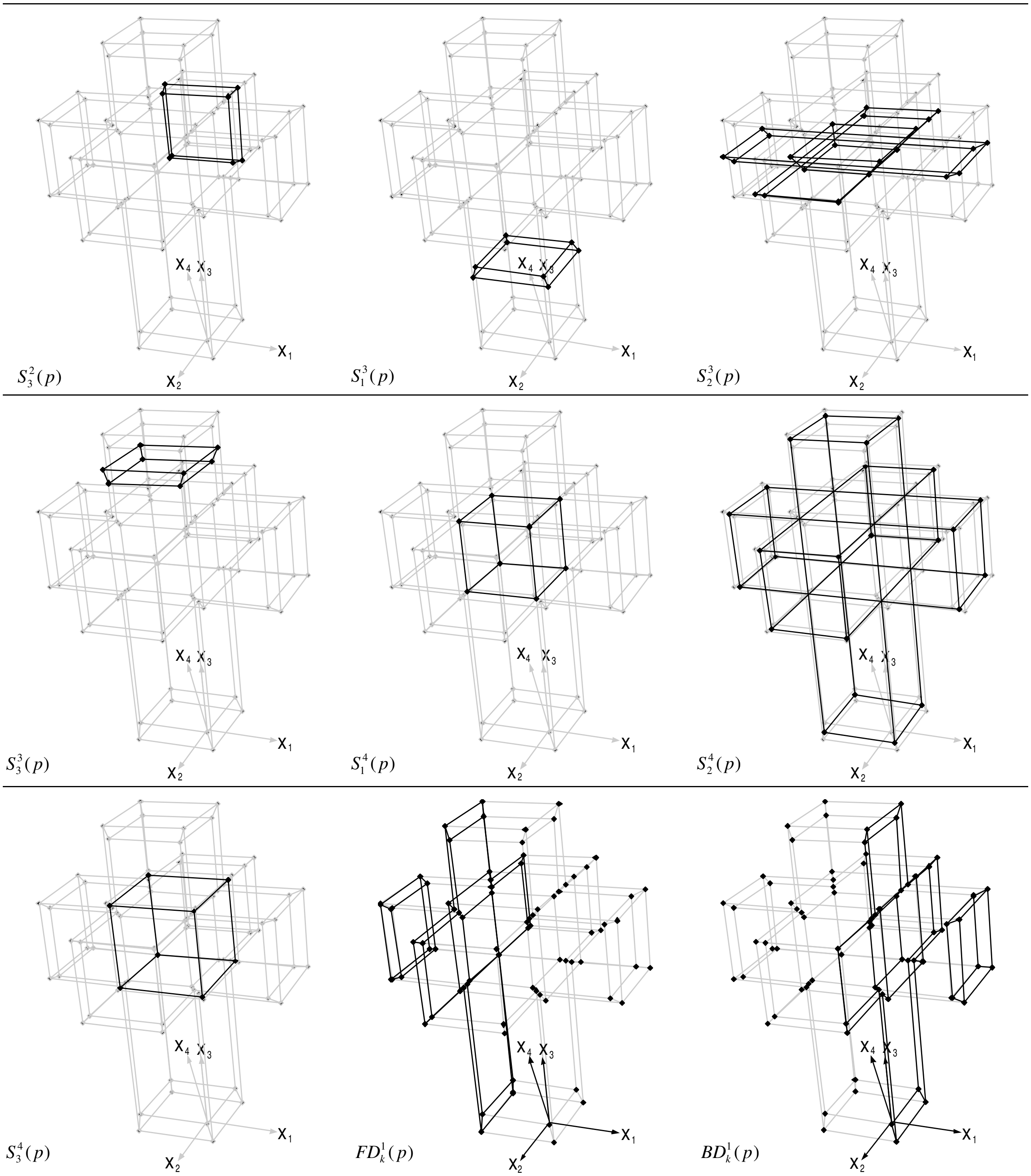


Table 5.18. Determining a 4D-OPP p from its 4D-EVM (Part 4, See text for details).

By applying **Corollary 5.8** we obtain internal Sections $S_1^2(p)$, $S_2^2(p)$ (**Table 5.17**) and $S_3^2(p)$ (**Table 5.18**) perpendicular to X_2-axis. In the same manner we obtain internal Sections perpendicular to X_3-axis ($S_1^3(p)$ to $S_3^3(p)$) and the internal Sections perpendicular to X_4-axis ($S_1^4(p)$ to $S_3^4(p)$). The 3D-EVM's for each $S_1^i(p)$ and $S_3^i(p)$, for all $1 \leq i \leq 4$, corresponds to 3D boxes while the 3D-EVM's for each $S_2^i(p)$, for all $1 \leq i \leq 4$, corresponds to solid 3D-crosses, i.e., tesseracts [Aguilera02b], as shown in **Figure 5.23.a**.

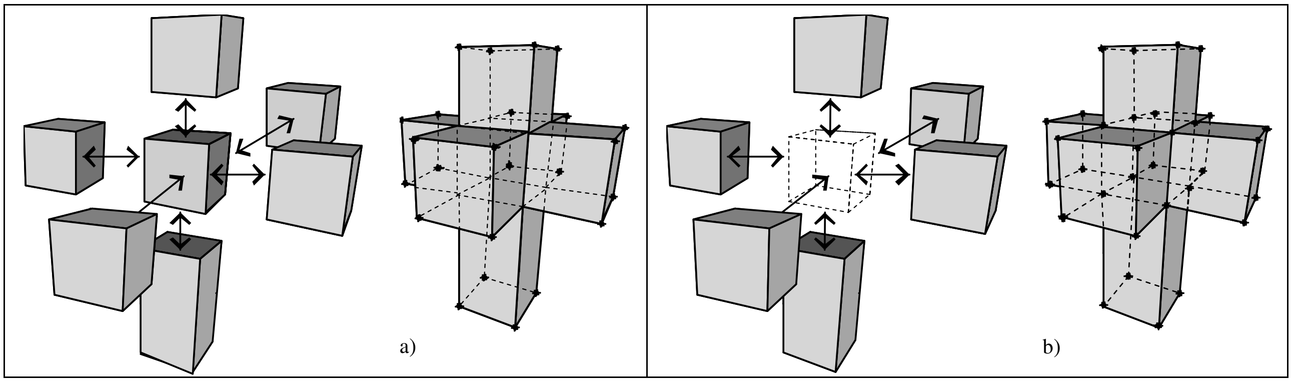


Figure 5.23. a) Composing a tesseract as the union of seven 3D-boxes; six of them share a face with a central box. In this case all its vertices are extreme except those present in the central box. This case corresponds to internal Sections $S_2^i(p)$, for all $1 \le i \le 4$, in **Tables 5.17** and **5.18**. b) Composing a 3D "cross" by removing the central box but maintaining the original dispositions of the remaining boxes. Because each vertex in the final 3D-OPP is surrounded by an odd number of boxes, each one of its vertices is extreme. This case corresponds to couplets $\Phi_2^i(p)$ and $\Phi_3^i(p)$, for all $1 \le i \le 4$, in **Tables 5.15**, **16** and **17**.

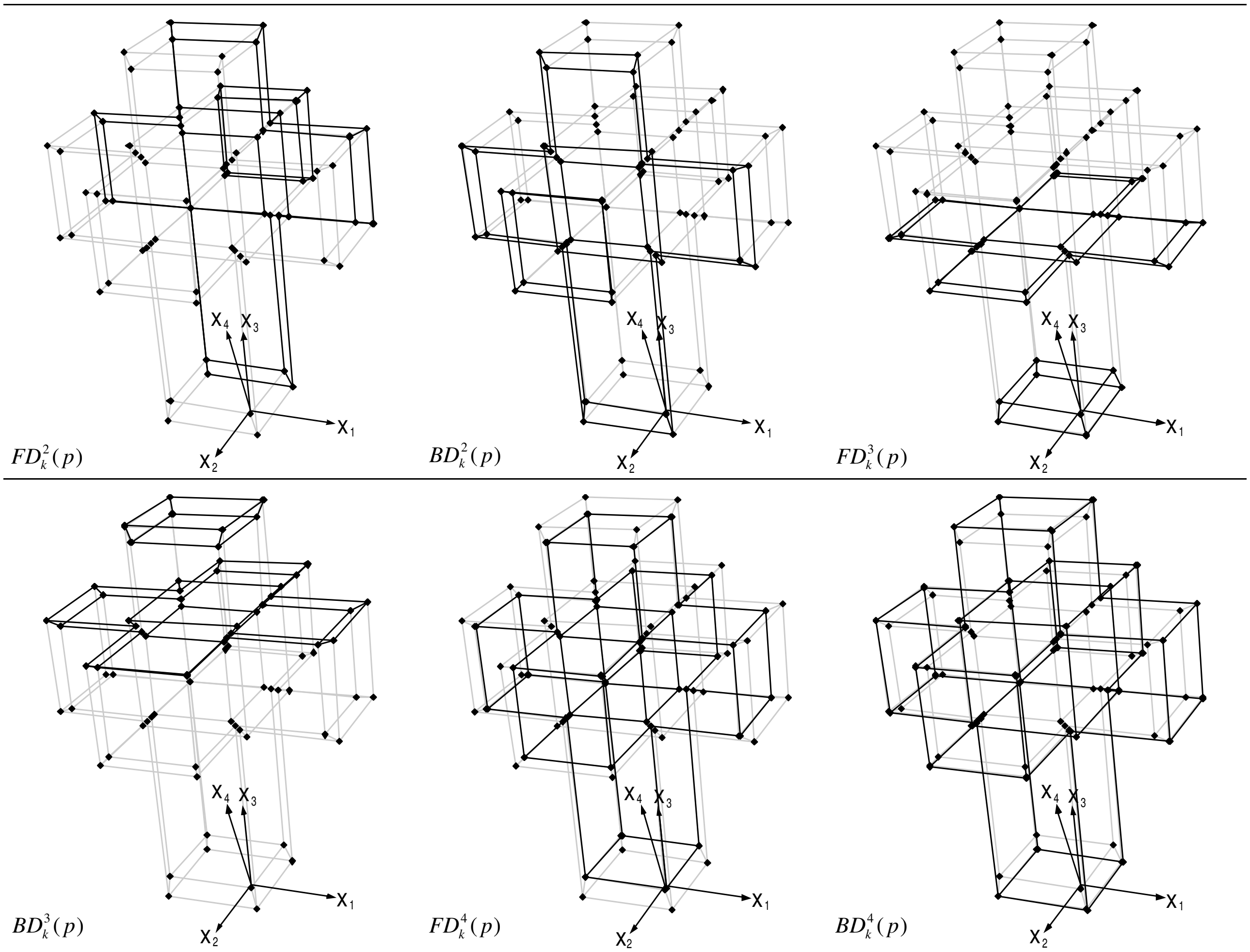


Table 5.19. Determining a 4D-OPP p from its 4D-EVM (Part 5, See text for details).

We obtain the correct orientation of each couplet by computing Forward and Backward differences as shown in **Table 5.19**. $FD_k^1(p)$ shows the couplets whose normal vector points towards the negative side of X_1-axis. $BD_k^1(p)$ shows the couplets whose normal vector points towards the positive side of X_1-axis. In similar manner we have that $FD_k^2(p)$, $FD_k^3(p)$ and $FD_k^4(p)$ are the couplets whose normal vector points towards the negative side of X_2, X_3 and X_4-axes respectively, while $BD_k^2(p)$, $BD_k^3(p)$ and $BD_k^4(p)$ are the couplets whose normal vector points towards the positive side of X_2, X_3 and X_4-axes respectively.

Through all the information obtained in **Tables 5.15** to **5.19,** and **Appendix G**, we can conclude that the represented 4D polytope corresponds to a 4D hyper-tesseract (a polytope that can be seen as the result of the unraveling of a 5D hypercube. See [Aguilera01] & [Aguilera02c] for more details).

At this point is interesting to consider an aspect that can be exemplified through **Figure 5.23**. According to **Property 5.1** even edges do not belong to brinks. An uninterrupted mixture of even and odd collinear contiguous edges can not be considered as a single brink. **Figure 5.23.a** shows the tesseract and due to the way it can be assembled all its edges are odd edges. Hence, all its extended edges contain only one brink. By removing the central box, in **Figure 5.23.b**, those edges which were previously adjacent to the removed central box are now characterized as even edges. Those extended edges passing through those even edges are now composed by two brinks.

5.8.2. Final Comments

In this chapter we have presented and analyzed the **Extreme Vertices Model in the n-Dimensional Space** (**nD-EVM**). The Extreme Vertices Model allows representing nD-OPP's by means of a single subset of their vertices: the *Extreme Vertices*. As commented in the beginning of this chapter itself, the new concept of *Odd Edge* has a paramount role in the fundamentals of the model. Since the works of Aguilera & Ayala, it is well known that although the EVM of an nD-OPP p has been defined as a subset of the nD-OPP's vertices, there is much more information about p hidden within this subset of vertices. This chapter has extended Aguilera & Ayala's techniques to the nD case in order to obtain this information:

- Computing sections from couplets.
- Computing couplets from sections.
- Computing forward and backward differences of consecutive sections.
- Computing the regularized XOR between two nD-OPP's represented through the nD-EVM.
- The conditions for a set of points in nD space to be a valid nD-EVM.

At this point it should be natural to ask about the behavior of the nD-EVM in the "real world". Next chapters will deal with the conversion of the nD-EVM from and to other polytopes representation schemes and with the development of efficient algorithms and their associated complexity.