Chapter 1
Introduction

Recent interest has been growing in studying multidimensional polytopes (4D and beyond) for representing multidimensional phenomena in the Euclidean n-dimensional space. Some of these phenomena's features rely on the polytope's geometric and topologic relations. In this sense, the Computer Science field has a very important relation with these studies. Moreover, Banchoff motivates us to think about two important questions [Banchoff96]:

1. **Is it possible to visualize a polytope to know how it looks like?**
2. **And if we can't see it, how can we be sure about the proper understanding of its relations and properties?**

The answers to provide to these questions should take in account that the task of visualizing and analyzing polytopes in the fourth and higher dimensions belongs to fields such as Computer Graphics, Computational Geometry, Polytopes Modeling, Topology and Geometry.

Answers for the first question have been given since 1966 with the computer's incursion to the field:

- At the Bell Labs, in 1966, Michael Nöll created the first computer images of 4D hypercubes. Important features of Nöll's programs were the use of stereo vision and 4D perspective projection ([Robbin92] and [Hollasch91]). The Nöll's method was the generation of the pictures via plotter and the transference onto film [Nöll67].
- Thomas Banchoff, at the Mathematics Department at Brown University, is one of the pioneers in the development of computer programs that allow interactive manipulation, for example with a joystick, of higher dimensional polytopes. Banchoff's technique of visualization is the projection of the polytopes' shadows onto 2D computer screens [Robbin92]. Banchoff and Charles Strauss are authors of the short film *"The Hypercube: Projections and Slicing"*, which was presented at the International Congress of Mathematicians in Helsinki in 1978.
- In [Hollasch91] it is mentioned the work of Scott Carey and Victor Steiner. They have rendered 4D polytopes to produce 3D "images", like the rendering of a 3D object produces a 2D image. Finally, the results of 4D-3D rendering are 3D voxel fields.
- Hollasch proposes a 4D ray-tracer that supports four-dimensional lighting, reflections, refractions, and also solves the hidden surfaces and shadowing problems in 4D space. The proposed ray-tracing method employs true four-space viewing parameters and geometry. Finally, the produced 3D field of RGB values is rendered with some of the existing methods [Hollasch91].
- In [Zhou91] is described a method for visualizing curves, surfaces and hyper-surfaces embedded in the 4D space. Such method is mainly based in the 4D-3D-2D projection technique. It presents methodologies for applying quaternions in the definition of rotations, and describes algorithms for polygonization of surfaces in detail. Finally, it demonstrates some geometric properties and phenomena characteristic of 4D space.
- Banks presents techniques for interaction with 4D-surfaces projected in the computer screen. Banks describes the ways to recover lost information that the 4D-3D-2D projection causes by means of visualization cues like depth. Also, ten degrees of freedom in 4D space are identified (6 rotations and 4 translations) and the use of devices to control the interaction, using all these freedom degrees, is described [Banks94].
- [Gunn93] describes a software implementation for visualizing 4D hyper-surfaces. Its application does not consider only the 4D Euclidean Space, because it is possible the visualization in 4D hyperbolic and spherical hyperspaces. Furthermore, it is possible to visualize the hyper-surfaces from an intrinsic point of view (with the observer embedded in the hyper-surface) or from an extrinsic point of view (with the observer outside the hyper-surface).
- Hanson et al coin the concept of *Visualizable Geometry*, which can be understood as the set of systems, concepts and methodologies related to the visualization of hyperspaces under distinct geometries and its applications [Hanson94].
- In [D'Zmura00] and [D'Zmura01] is described the *Hyper* system, which was developed by researchers at the Department of Cognitive Sciences at the University of California, Irvine. The system's main objective is the creation of 4D virtual worlds and the users' interaction through Virtual Reality devices. Moreover, they describe experiments where the users are randomly positioned in a 4D virtual world (which was generated by

Hyper system). The users must find a target in the minor possible time. Each user repeated the experiment several times. By the obtained results, they conclude that an individual is able to navigate efficiently in environments with four dimensions after some sessions of training.
- Aguilera & Pérez-Aguila, in [Aguilera01], [Aguilera02b] and [Aguilera02c], discuss the method for visualizing 4D polytopes through their *unravelings* and present methodologies for unraveling the hypercube and the 4D simplex.
- In [Kolcun04] it is introduced the interpretation of 3D oriented volumes in order to generalize, to the 4D case, the visibility criteria for convex polytopes.
- Julieta C. Aguilera, at the Electronic Visualization Laboratory in the University of Illinois at Chicago, has designed a virtual reality application where interaction with a four dimensional structure can be facilitated by using body motion. In this application, the degrees of freedom of an arm and hand are used to represent the projection of a 4D rotation, and relate the resulting form to the natural experience and intention of the user. According to her results, the space becomes an adaptive structure tied to the structure of the user's body which allows him/her to inhabit higher dimensional spaces via computer graphics ([Aguilera06] & [Aguilera06b]).

What about Banchoff's second question? In first place we can say that it is well known that Solid Modeling is an area of wide development in several applications as the Computer Aided Design and Manufacturing (CAD/CAM), electronic prototypes, animation planning, etc. If a 3D solid object can be modeled in a way that its geometry is appropriately captured, then it will be possible to apply, on such object, a range of useful operations. Due to the need of modeling objects as solids, the development of a variety of specialized mechanisms to represent them has arisen. The representation schemes for solid objects are frequently divided in some large categories (although not all the representations are completely inside in one of them): Boundary Representations, Spatial Partitioning Representations, Constructive Solid Geometry, etc. The extensions of the solid modeling schemes, by considering their application to spaces beyond the three-dimensional, have allowed the modeling of n-dimensional polytopes [Paoluzzi93]. Some schemes for the Polytopes Modeling are usually concentrated on two categories:

- The n-Dimensional Boundary Representations: They describe an nD polytope in terms of the elements that compose its boundary and these representations have the information about the connectivity between these elements.
- Hyperspatial Partitioning Representations: Where a polytope is decomposed in a collection of attached n-dimensional cells, without intersections, and more primitive than the original polytope, although they are not necessarily of the same kind. Inside this category we can find schemes as the n-Dimensional Cell Decompositions, the Hypervoxelizations and the 2^n-trees (hyperoctress).

A new adding to the answer for Banchoff's second question was given in 1997. The Extreme Vertices Model (3D-EVM) was originally presented, and widely described, by Aguilera & Ayala for representing 2-manifold Orthogonal Polyhedra [Aguilera97] and later considering both Orthogonal Polyhedra (3D-OP's) and Pseudo-Polyhedra (3D-OPP's) [Aguilera98]. This model has enabled the development of simple and robust algorithms for performing the most usual and demanding tasks on solid modeling, such as closed and regularized Boolean operations, solid splitting, set membership classification operations and measure operations on 3D-OPP's. As mentioned in [Aguilera98], it is natural to ask if the EVM can be extended for modeling n-Dimensional Orthogonal Pseudo-Polytopes (nD-OPP's). In this sense, some experiments have been made, by Pérez-Aguila & Aguilera [Pérez-Aguila03d], where the validity of the model was assumed true in order to represent 4D and 5D-OPPs. The results obtained have leaded us to state the **Main Hypothesis** of this work:

The Extreme Vertices Model in the n-Dimensional Space (nD-EVM) is a complete scheme for the representation of n-Dimensional Orthogonal Pseudo-Polytopes.

The meaning of complete scheme is based in Requicha's set of formal criterions that every scheme must have rigorously defined: Domain, Completeness, Uniqueness and Validity [Requicha80].

The spirit behind the work we will develop in the next chapters is to share a little contribution to the answer for Banchoff's question two. The purpose of this work is to prove in a purely formal and systematical way that the Extreme Vertices Model allows representing and manipulating nD-OPP's by means of a single subset of their vertices: the Extreme Vertices. It will be seen how the Odd Edge Combinatorial Topological Characterization in the nD-OPP's has a paramount role in the foundations of the nD-EVM. Although the EVM of an nD-OPP has been defined as a subset of the nD-OPP's vertices, there is much more information about the polytope hidden within this subset of vertices. We will show the basic procedures and algorithms in order to obtain this information.

Besides this chapter, the structure of this document is the following:

- **Chapter 2 - Theoretical Frame and Previous Work:** This chapter will introduce some results and concepts which are related with the study to be described in this work. We will focus in basic definitions related to the dominion of the objects we consider in our research. Moreover, it describes some schemes for the Modeling of n-Dimensional Polytopes. We will briefly comment the n-Dimensional Boundary Representations, Hypervoxelizations, 2^n-trees and the n-Dimensional Simplexation of Convex Polytopes. We will summarize our previous work which is related to the topological characterization of the elements that compose the boundary of the 4D Orthogonal Pseudo-Polytopes.

- **Chapter 3 – Configurations in the n-Dimensional Orthogonal Pseudo-Polytopes:** This chapter will describe some relations and strategies that could support us in the task of obtaining in a more direct way these configurations. In order to speed up the determination of the topological equivalence between a pair of configurations, we describe relations whose implementation compares any two configurations in a time which only depends of the number of hyper-octants in the space in which their hyper-boxes are embedded. We will show that our relations are in fact equivalence relations which are 'wider' than the classical equivalence relation based in geometrical transformations and therefore they provide an approximate solution to our problem.

- **Chapter 4 – The Odd Edge Characterization and its Role in the Combinatorial Topology of the n-Dimensional Orthogonal Pseudo-Polytopes:** This chapter is related directly with Local Analysis over the nD-OPP's. The results obtained eventually will lead us to the formalization of the Extreme Vertices Model in the n-Dimensional Space. We will introduce formally the notion of Odd Edge, in a combination of nD hyper-boxes, as an edge with an odd number of incident hyper-boxes. We will analyze how this very simple concept provides us important information about the combinatorial nature of the nD-OPP's from a topological point of view. The properties we obtain will perform an important and essential role when we define the foundations behind the Extreme Vertices Model.

- **Chapter 5 – Orthogonal Polytopes Modeling Through the Extreme Vertices Model in the n-Dimensional Space (nD-EVM):** This chapter represents the kernel of this work because in it we will prove our **Main Hyphotesis**. We will establish the foundations behind the nD-EVM. It will be seen how the Odd Edge Topological Characterization in the nD-OPP's has a paramount role in this last aspect. We will deal with Local and Global Analysis over the nD-OPP's but now under the context of the nD-EVM. Finally, the concepts and results originally presented by Aguilera & Ayala will be presented and discussed under the new context of the nD-EVM.

- **Chapter 6 – Algorithms in the nD-EVM and their Performance:** In this chapter we will introduce some basic algorithms that perform operations between nD-OPP's represented through the nD-EVM. We will specify and study aspects related to their implementation and their temporal complexity. We will deal with basic algorithms to be considered for the manipulation of nD-EVM's. Details about the way we are storing and implementing EVM's will be provided. The algorithms we describe and test experimentally provide solutions in order to perform Regularized Boolean Operations, computing of nD Content, computing of (n-1)D Content, and extraction of boundary elements of nD-OPP's represented through the nD-EVM.

- **Chapter 7 – Applications:** Where we will consider five applications which are developed under the context of the nD-EVM: 1) Classification of Color 2D-Images, 2) Representation and manipulation of Color 2D and 3D Animations, 3) Collision detection of 3D objects, 4) Enhancing Image Based Reasoning, and 5) Manipulation of "real world" 3D datasets. We will describe the way the algorithms and operations defined under the nD-EVM can be applied in order to solve the required tasks by these applications.

- **Chapter 8 – Conclusions and Future Work:** Where the main contributions of this work are summarized and the identified perspectives of future research are established.