

Appendix J

Properties of ρ Function

Definition 7.1 [Pérez-Aguila05c]: Let $x, y \in \mathbb{R}^+$. Let ρ be the function described as

$$\rho(x, y) = \begin{cases} 1 - \frac{x}{y} & \text{if } x < y \\ 1 - \frac{y}{x} & \text{if } y < x \\ 0 & \text{if } x = y \end{cases}$$

Property J.1: Let $x, y \in \mathbb{R}^+$. Therefore $\rho(x, y) \in [0, 1]$.

Proof:

We consider three cases,

- If $x = y \Rightarrow \rho(x, y) = 0$.
- If $x < y \Rightarrow 1 > x/y > 0 \Rightarrow -1 < -x/y < 0 \Rightarrow 0 < 1 - x/y < 1 \Rightarrow 0 < \rho(x, y) < 1$.
- If $x > y \Rightarrow 1 > y/x > 0 \Rightarrow -1 < -y/x < 0 \Rightarrow 0 < 1 - y/x < 1 \Rightarrow 0 < \rho(x, y) < 1$.

$$\therefore (\forall x, y \in \mathbb{R}^+) (\rho(x, y) \in [0, 1])$$

□

Lemma J.1: Let $x, y, z \in \mathbb{R}^+$ such that $x < y < z$. Then $\rho(x, z) < \rho(x, y) + \rho(y, z)$.

Proof:

By **Definition 7.1** and considering the established hypothesis we have that

$$\rho(x, y) = 1 - x/y \quad \rho(y, z) = 1 - y/z \quad \rho(x, z) = 1 - x/z$$

Because by hypothesis, $x < y \Rightarrow x/z < y/z \Rightarrow -x/z > -y/z \Rightarrow 1 - x/z > 1 - y/z \Rightarrow \rho(x, z) > \rho(y, z)$.

Because by hypothesis, $y < z \Rightarrow x/y > x/z \Rightarrow -x/y < -x/z \Rightarrow 1 - x/y < 1 - x/z \Rightarrow \rho(x, y) < \rho(x, z)$.

Due to $1 > \rho(x, z) > \rho(y, z)$ and $1 > \rho(x, z) > \rho(x, y)$

$$\Rightarrow 2 > 2\rho(x, z) > \rho(x, y) + \rho(y, z) = 2 - (x/y + y/z) \Rightarrow 2 > 2 - (x/y + y/z)$$

$$\Rightarrow 1 > 1 - (x/y + y/z) \Rightarrow 2 > \rho(x, y) + \rho(y, z) > 1 > 1 - (x/y + y/z)$$

$$\Rightarrow \rho(x, y) + \rho(y, z) > 1 \Rightarrow \rho(x, y) + \rho(y, z) > 1 > \rho(x, z).$$

$$\therefore (\forall x, y, z \in \mathbb{R}^+) (x < y < z) (\rho(x, z) < \rho(x, y) + \rho(y, z))$$

□

Lemma J.2: Let $x, y, z \in \mathbb{R}^+$ such that $x < z < y$. Then $\rho(x, z) < \rho(x, y) + \rho(y, z)$.

Proof:

By **Definition 7.1** and considering the established hypothesis we have that

$$\rho(x, y) = 1 - x/y \quad \rho(y, z) = 1 - y/z \quad \rho(x, z) = 1 - x/z$$

Because by hypothesis, $x < z \Rightarrow x/y < z/y \Rightarrow -x/y > -z/y \Rightarrow 1 - x/y > 1 - z/y \Rightarrow \rho(x, y) > \rho(y, z)$.

Because by hypothesis, $z < y \Rightarrow x/z > x/y \Rightarrow -x/z < -x/y \Rightarrow 1 - x/z < 1 - x/y \Rightarrow \rho(x, z) < \rho(x, y)$.

Due to $\rho(x, z) < \rho(x, y)$ and $\rho(y, z) < \rho(x, y)$

$$\therefore (\forall x, y, z \in \mathbb{R}^+) (x < z < y) (\rho(x, z) < \rho(x, y) + \rho(y, z))$$

□

Lemma J.3: Let $x, y, z \in \mathbb{R}^+$ such that $z < x < y$. Then $\rho(x, z) < \rho(x, y) + \rho(y, z)$.

Proof:

By **Definition 7.1** and considering the established hypothesis we have that

$$\rho(x, y) = 1 - x/y \quad \rho(y, z) = 1 - y/z \quad \rho(x, z) = 1 - z/x$$

Because by hypothesis, $x < y \Rightarrow z/x > z/y \Rightarrow -z/x < -z/y \Rightarrow 1 - z/x < 1 - z/y \Rightarrow \rho(x, z) < \rho(y, z)$

$$\Rightarrow \rho(x, z) < \rho(y, z) < \rho(x, y) + \rho(y, z)$$

$$\therefore (\forall x, y, z \in \mathbb{R}^+) (z < x < y) (\rho(x, z) < \rho(x, y) + \rho(y, z))$$

□

Lemma J.4: Let $x, y, z \in \mathbb{R}^+$ such that $z < y < x$. Then $\rho(x, z) < \rho(x, y) + \rho(y, z)$.

Proof:

By **Definition 7.1** and considering the established hypothesis we have that

$$\rho(x, y) = 1 - y/x \quad \rho(y, z) = 1 - z/y \quad \rho(x, z) = 1 - z/x$$

Because by hypothesis, $z < y \Rightarrow z/x < y/x \Rightarrow -z/x > -y/x \Rightarrow 1 - z/x > 1 - y/x \Rightarrow \rho(x, z) > \rho(x, y)$.

Because by hypothesis, $y < x \Rightarrow z/y > z/x \Rightarrow -z/y < -z/x \Rightarrow 1 - z/y < 1 - z/x \Rightarrow \rho(y, z) < \rho(x, z)$.

Due to $1 > \rho(x, z) > \rho(y, z)$ and $1 > \rho(x, z) > \rho(x, y)$

$$\Rightarrow 2 > 2\rho(x, z) > \rho(x, y) + \rho(y, z) = 2 - (y/x + z/y) \Rightarrow 2 > 2 - (y/x + z/y)$$

$$\Rightarrow 1 > 1 - (y/x + z/y) \Rightarrow 2 > \rho(x, y) + \rho(y, z) > 1 > 1 - (y/x + z/y)$$

$$\Rightarrow \rho(x, y) + \rho(y, z) > 1 \Rightarrow \rho(x, y) + \rho(y, z) > 1 > \rho(x, z)$$

$$\therefore (\forall x, y, z \in \mathbb{R}^+)(z < y < x)(\rho(x, z) < \rho(x, y) + \rho(y, z)) \quad \square$$

Lemma J.5: Let $x, y, z \in \mathbb{R}^+$ such that $y < x < z$. Then $\rho(x, z) < \rho(x, y) + \rho(y, z)$.

Proof:

By **Definition 7.1** and considering the established hypothesis we have that

$$\rho(x, y) = 1 - y/x \quad \rho(y, z) = 1 - z/y \quad \rho(x, z) = 1 - z/x$$

Because by hypothesis, $y < x \Rightarrow y/z < x/z \Rightarrow -y/z > -x/z \Rightarrow 1 - y/z > 1 - x/z$

$$\Rightarrow \rho(y, z) > \rho(x, z) \Rightarrow \rho(x, z) < \rho(y, z) < \rho(x, y) + \rho(y, z)$$

$$\therefore (\forall x, y, z \in \mathbb{R}^+)(y < x < z)(\rho(x, z) < \rho(x, y) + \rho(y, z)) \quad \square$$

Lemma J.6: Let $x, y, z \in \mathbb{R}^+$ such that $y < z < x$. Then $\rho(x, z) < \rho(x, y) + \rho(y, z)$.

Proof:

By **Definition 7.1** and considering the established hypothesis we have that

$$\rho(x, y) = 1 - y/x \quad \rho(y, z) = 1 - z/y \quad \rho(x, z) = 1 - z/x$$

Because by hypothesis, $y < z \Rightarrow y/x < z/x \Rightarrow -y/x > -z/x \Rightarrow 1 - y/x > 1 - z/x$

$$\Rightarrow \rho(x, y) > \rho(x, z) \Rightarrow \rho(x, z) < \rho(x, y) < \rho(x, y) + \rho(y, z)$$

$$\therefore (\forall x, y, z \in \mathbb{R}^+)(y < z < x)(\rho(x, z) < \rho(x, y) + \rho(y, z)) \quad \square$$