Appendix C
Some Adjacencies' Properties of the Configurations in the nD-OPP's

The following propositions were presented originally in [Aguilera04].

Theorem C.1: *The number of adjacencies for any combination with x hyper-boxes is:*

$$\frac{x(x-1)}{2} = \frac{1}{2}\left(x^2 - x\right)$$

Proof:
A first hyper-box of the configuration will have *x-1* adjacencies (one for each *x-1* hyper-boxes); a second hyper-box will have *x-2* adjacencies (not including the adjacency with the first hyper-box because it is in that first hyper-box counting); a third hyper-box will have *x-3* adjacencies (not including the adjacencies with the first and second hyper-boxes because they are in these hyper-boxes' respective counting); in general, a k-th hyper-b*ox* (*k* < *x*) will have *x-k* adjacencies. The adjacencies' total counting (i.e. the sum of all hyper-boxes' adjacencies) is defined by the well known expression for the sum of the first *x-1* positive integers:

$$\sum_{k=1}^{x-1} k = \frac{x(x-1)}{2} = \frac{x^2 - x}{2}$$

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Observation C.1: *In a n-dimensional configuration consider a m-dimensional subspace ($0 \le m < n$) that passes through the origin. The maximum number of adjacencies embedded in that m-dimensional subspace is 2^{n-1}.*

Lemma C.1: *In the n-dimensional space, the maximum number of m-dimensional adjacencies for the configuration with 2^n boxes (the configuration with a hyper-box in all its hyper-octants) is:*

$$\binom{n}{m} \cdot 2^{n-1}, \qquad 0 \le m < n$$

Proof:
(n,m) is the number of *m*-dimensional subspaces, which are composed by the *m* axes from the n-dimensional space, and there are 2^{n-1} *m*-dimensional adjacencies for each one (by Observation C.1). ◪

Corollary C.1: *The total number of adjacencies in a configuration with 2^n hyper-boxes (the configuration with a hyper-box in all its hyper-octants) is:*

$$\sum_{m=0}^{n-1} \binom{n}{m} \cdot 2^{n-1}$$

Proof:
Each one of its terms will provide the number of *m*-dimensional adjacencies for the configuration with 2^n boxes. The upper limit for *m* is *n-1* since $0 \le m < n$ (see Observation C.1). ◪

Corollary C.2: *The sum of adjacencies for the n-dimensional configuration with 2^n hyper-boxes (i.e., with all its hyper-octants filled) is:*

$$\frac{1}{2}\left(2^{2n} - 2^n\right)$$

Proof:
Theorem C.1 provides a formula for the sum of adjacencies in a configuration with *x* boxes: $(x^2 - x)/2$. By letting $x=2^n$ then the sum of all adjacencies for the configuration with all its hyper-octants filled will be obtained:

$$\frac{1}{2}\left(\left(2^n\right)^2 - \left(2^n\right)\right) = \frac{1}{2}\left(2^{2n} - 2^n\right)$$

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Theorem C.2: *A closed form for evaluating the sum in Corollary C.1 is given by Corollary C.2*

$$\sum_{m=0}^{n-1}\binom{n}{m}\cdot 2^{n-1}=\frac{1}{2}\left(2^{2n}-2^{n}\right)$$

Proof:

It is well known that $\sum_{m=0}^{n}\binom{n}{m}=2^{n}$ and since $\binom{n}{n}=1$, then $\sum_{m=0}^{n-1}\binom{n}{m}=2^{n}-1$. Therefore:

$$\sum_{m=0}^{n-1}\binom{n}{m}\cdot 2^{n-1}=2^{n-1}\cdot(2^{n}-1)=\frac{1}{2}\left(2^{2n}-2^{n}\right)$$

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Corollary C.3: *The total number of adjacencies in a configuration with 2^n-1 hyper-boxes is:*

$$\sum_{m=0}^{n-1}\binom{n}{m}\cdot(2^{n-1}-1)$$

Proof:

We know by Observation C.1 and Lemma C.1 that there are at most 2^{n-1} adjacencies in each one of the possible *(n,m)* m-dimensional subspaces in the configuration with 2^n hyper-boxes. By removing a hyper-box from this configuration we remove an adjacency in each one of these m-dimensional subspaces. ◪

Corollary C.4: *The sum of adjacencies for the n-dimensional configuration with 2^n-1 hyper-boxes is:*

$$(2^{n}-1)(2^{n-1}-1)=2^{2n-1}-2^{n}-2^{n-1}+1$$

Proof:

Theorem C.1 provides a formula for the sum of adjacencies in a configuration with *x* hyper-boxes: $(x^2-x)/2$. By letting $x=2^n-1$ then the sum of all adjacencies for the configuration with 2^n-1 hyper-boxes will be obtained:

$$\frac{1}{2}\left(\left(2^{n}-1\right)^{2}-\left(2^{n}-1\right)\right)=2^{2n-1}-2^{n}-2^{n-1}+1$$

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Theorem C.3: *A closed form for evaluating the sum in Corollary C.3 is given by Corollary C.4*

$$\sum_{m=0}^{n-1}\binom{n}{m}\cdot(2^{n-1}-1)=2^{2n-1}-2^{n}-2^{n-1}+1$$

Proof:

It is well known that $\sum_{m=0}^{n}\binom{n}{m}=2^{n}$ and since $\binom{n}{n}=1$, then $\sum_{m=0}^{n-1}\binom{n}{m}=2^{n}-1$. Therefore:

$$\sum_{m=0}^{n-1}\binom{n}{m}\cdot(2^{n-1}-1)=(2^{n-1}-1)(2^{n}-1)=2^{2n-1}-2^{n}-2^{n-1}+1$$

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